Angles, trigonometric functions, and university level Analysis

Carl Winsløw[1],

[1]University of Copenhagen, Faculty of Science, Denmark, winslow@ind.ku.dk

The transition from calculus to analysis presents well known challenges to many students. In line with Chevallard's "paradigm of questioning the world", we consider that part of the difficulty is a failure to question (explore) what the formal theory of analysis means for "familiar objects" from calculus. As a significant case, we consider the familiar notions of "angle", cosine and sine. We first examine some "advanced" viewpoints on these objects coming from analysis, then the viewpoints an advanced student could produce on these notions, based on knowledge from an analysis course in which resources for a rigorous account were indeed available.

Keywords: calculus, analysis, angles, sine, cosine.

INTRODUCTION: QUESTIONING IN UNDERGRADUATE ANALYSIS

This paper is motivated by observations and reflections on the calculus-analysis transition which we have studied for several years (Gyöngyösi, Solovej, & Winsløw, 2011; Winsløw, 2008). In particular, we have observed that students do not easily relate the contents of analysis courses (formal theory of continuity, integration, function spaces etc.) to *calculus* as encountered in high school and the first few courses at university. Even evident relations, such as the use of concrete cases to question general assertions on continuity or convergence of various types, are far from easy to establish; for many students, the more abstract theory seems to constitute a universe in itself. Our original motivation for this problem was the hypothesis that supporting students' work with such "concrete-abstract" relations could be used to prevent some of the widespread failure which students experience in the more abstract courses. In (Winsløw & Grønbæk, 2014) we investigated the potentials and obstacles of the opposite transition, as pursued in so-called *capstone* courses for future teachers. Finally, in (Winsløw, 2016) we investigated the potential of exercises explicitly *questioning* key definitions in the setting of abstract courses. The didactical significance of "questioning" was further developed, based on (Chevallard, 2012); the idea is for students to meet and develop some of the more fundamental questions which gave rise to an otherwise unmotivated abstract definition, such as the definition of uniform continuity of functions; such a questioning often leads to recover relations between the abstract theory with classical and concrete problems, and in particular, to consider the role of definitions within a theory - indeed, that they are not mere arbitrary conventions, but in fact often essential elements of crucial mathematical advances.

In the present paper, we investigate a concrete case which touches upon all three aspects: the relation *R* between *the problem of giving a rigorous definition of sine*, and *the material covered in a first rigorous course on analysis* (including, as a minimum, a theory of integration of functions on an interval and along a rectifiable curve in the plane). We begin with a historical background section, referring essentially to Klein's ideas on R and similar relations (Klein, 1908/1932). We proceed to a study of how this problem is treated in two textbooks for analysis courses of the type mentioned above, the aim being to identify the relations which such courses could aim to develop. Finally, we report on an interview with an advanced and successful student of mathematics who, in particular, served as a teaching assistant on a course of the type mentioned, in view of identifying an "upper bound" of the relations of type R which typical students in that course could develop.

BACKGROUND: PLAN A AND PLAN B

Classical analysis can be described as the study of *functions*, with an emphasis on so-called elementary functions, of which the most important are polynomials, power functions $x \mapsto x^a$, exponential functions $x \mapsto a^x$, the logarithmic functions $\log_a$, and trigonometric functions (chiefly sin and cousins). Felix Klein (1908/1932, pp. 77-85) considered that in the history of mathematics, as well as the school discipline, we may identify two possible "Plans" (one might also say, visions or strategies) for developing a subject; and he used the construction and study of elementary functions as a main case to illustrate these plans.

What Klein calls *Plan A* is a *compartmentalized approach* to mathematics, which emphasizes precise and purified work within certain small "areas" of mathematics, which are hardly related among each other:

> Plan A is based upon a more particularistic conception of science which divides the total field into a series of mutually separated parts and attempts to develop each part for itself, with a minimum of resources and with all possible avoidance of borrowing from neighbouring fields (ibid., p. 78).

For the case of the elementary functions, this corresponds to constructing and studying different classes of functions separately. For instance, the general exponential function $x \mapsto a^x$ is constructed based on progressive extension in terms of x (from integers to real exponents), taking care of explaining all details in the process. In another compartment, namely analytic geometry, one studies trigonometric functions in relation to triangles and circles. Naturally, in school versions of this plan, the level of detail and precision may vary. In any case, the two functions classes arise from quite distant universes. It is only students who continue to study mathematics at university who will learn of an almost "magical" relation among them, when the complex versions of these functions appear (and with them, Euler's formula $e^{ix} = \cos x + i \sin x$). The most famous examples of "Plan A" in the history of mathematics itself include, of course, Euclid's elements, and more generally the axiomatic method with its massive influence in all of modern mathematics.

Plan B, by contrast, involves a more holistic approach which emphasises and exploits connections between different sectors, sometimes at the expense of strict rigor:

> … the supporter of Plan B lays the chief stress upon the organic combination of the partial fields, and upon the stimulation which these exert one upon another. He prefers, therefore, the methods which open for him an understanding of several fields under a uniform point of view. His ideal is the comprehension of the sum total of mathematical science as a great connected whole (ibid., p. 78).

Klein observes that in the history of mathematics, “Plan A” and “Plan B” both appear during fruitful periods of research, in analysis as well as in other areas. For instance, the initial developments of the calculus took place much according to Plan B, led by Leibniz and Newton; later, a progressive move towards Plan A occured, as Cauchy and others gave classical analysis the solid foundations we know today.

Klein strongly recommends including “Plan B” as a strategy for presenting mathematics to students, and laments the exclusive use of Plan A at school (in his days). We note, as an aside which we will not further treat in this paper, that Klein also briefly mentions a “Plan C”: essentially, “pursuit of algorithms”. It could be considered as vastly more important today as in the days of Klein, which had no computers to fuel it; but we leave Plan C here, as it appears less relevant to the teaching analysis in a university program on pure mathematics. In such programs in general, Plan A prevails to the extent they consist of specialised modules on a few specific sectors, with little explicit links among them or to mathematics at large.

PLAN B IN CALCULUS

An introduction of exponential and trigonometric functions following Plan B could, according to Klein (ibid., pp. 155-169), be based on what he calls ‘quadratures of simple curves’ - in modern terms, functions given by definite integrals, and their inverses. It thus requires a solid calculus background. In outline, his proposal is:

$$\ln(x) = \int_1^x \frac{dt}{t};\ \exp(x) = \ln^{-1}(x)\ ;\ a^x = \exp(x\ln(a))$$

$$\arcsin(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}};\ \sin(x) = \arcsin^{-1}(x);\ \ \cos(x) = \sin(x + \arcsin(1)\ ).$$

Klein points out many analogies and relations between the two cases. Note, however, that one specific difficulty remains in the second case: to prove that the integral converges at ± 1. It can be solved using a limit argument on the identity (ibid., p. 168)

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} = 2\int_0^x \sqrt{1-t^2}dt - x\sqrt{1-x^2} \quad (-1 < x < 1)$$

as the right hand side is clearly meaningful when $x = \pm 1$.

The definition of arcsin could also be related to a formula encountered (and perhaps to some extent explained) in calculus classes, concerning the length of the graph of a function between two points on the graph. Using it on $f(y) = \sqrt{1-y^2}$ for $y \geq 0$, one finds that the length of the graph from (1,0) to $(f(y), y)$ is exactly $\arcsin(y)$ as defined above. We can therefore interpret $\arcsin(y)$ for $y \geq 0$ as the length of a unit circle arc from (1,0) to the unique point (x, y) on the unit circle with $x \geq 0$. Besides explaining the strange name “arc sine”, it also shows that this function gives a kind of “angle” of circle points (in the first quadrant directly, then by extension). Moreover, sine takes the “angle” ($\arcsin(y)$) to the second coordinate (y) of the point on the circle that is situated at that angle, as explained in a common school approach.

PLAN A IN SECONDARY SCHOOL

Klein’s approach is certainly not common in secondary school. Instead, various Plans A appear for the case of introducing cosine and sine, with three distinct contexts (Demir & Heck, 2013, p. 120):

- First, trigonometry appears as a “toolbox” for solving triangle problems in plane geometry, where cosine and sine are defined as certain ratios of sides in a right triangle, accompanied with something like Figure 1;
- Then, in the setting of analytic geometry, cosine and sine are defined as coordinates of the intersection of a ray through the unit circle, accompanied with something like Figure 2;
- Finally, cosine and sine emerge as functions through tables and graphs like Figure 3, with a discussion of function properties such as domain, range, zeros, period etc.

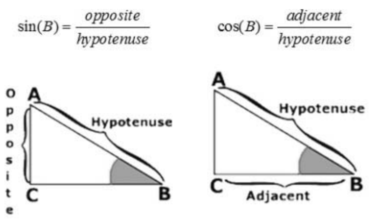

Figure 1: Triangle context.

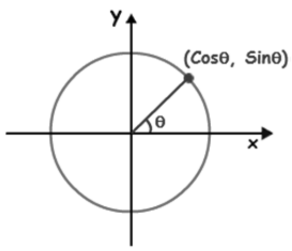

Figure 2: Cartesian context.

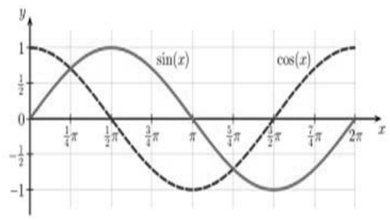

Figure 3: Function context.

Naturally, some efforts are being made to relate the three contexts, which usually appear in the order of the figures; elements of plan B can be seen in such efforts. We should note here that while Figure 1 and 2 are really used as central ostensives for different definitions (which are consistent, as is proved by appealing to similarity), Figure 3 is not presented as a definition, but more as an illustration of the definition shown in Figure 2; the postulate character of the graph is the main obstacle addressed by Demir and Heck (2013). No doubt, the graphs are instrumental - along with the symbolic ostensives, tables, and the use of usual terminology related to functions - to institutionalize the “function status” of cosine and sine. But some mystery remains:

> The sine and cosine functions may have been defined, but the graphs of these real functions remain mysterious or merely diagrams produced by a graphing calculator or mathematics software. The complex nature of trigonometry makes it challenging for students to understand the topic deeply and conceptually. (Demir & Heck, 2013, p. 119)

While we do agree with the authors that difficulties remain, a “deep understanding” may not result from simply multiplying the ostensives produced with dynamic software, since none of the “definitions” given are really basing the functions on firm mathematical ground. In fact they ultimately relate on the meaning of *angle*, naturalised since primary school as a “measure” of the space between two crossing line segments (for instance, sides in a triangle). Among the mysterious operations which usually accompany the passage from the triangle context to the Cartesian context is the unmotivated change of “unit” for this “measure”, as “degrees” are replaced by “radians”. The historic reasons for the appearance and rejection of the bizarre convention to associate, for instance, the magnitude 90^0 with a right angle, are certainly interesting, and there is no shortage of other units and notations which are or have been commonly used for “angles”. But whether or not these variations (or the sudden passage to radians) are questioned, they only distract from the heart of the matter: the meaning(fullness) of “angle measure”. It remains a mere postulate, with no mathematical grounds, that a real number may be associated to *any* pair of crossing lines, as somehow a “measure” of the “width” of the “space” between them. Of course, turning to Klein’s proposal (for secondary school) may indeed put trigonometric functions on a firmer basis, assuming (as he did) that integrals are given a more than superficial treatment at that level. However, this may at best rescue the functions as such, not the geometric interpretation shown in Figure 1 and 2, and in particular the more fundamental notion of angle measure. In fact, angle measure is nothing else than arc length in the special case of a unit circle, and just as the notion of area, it remains materially or sensually based in school for the simple reason that any mathematical definition depends on the topology of the real numbers.

GERMS OF PLAN B IN UNIVERSITY ANALYSIS

Standard calculus courses maintain, and do not question, the material approach to angles and trigonometric functions. Based on two different but typical texts from typical first courses on real analysis, we shall now see that in such courses, a deeper questioning is at least possible. This means that we may seek new meanings and explanations of two claims which are both plausible and familiar to the students:

(I) every point on the unit circle S^1 corresponds uniquely to a real number (“angle”) and these numbers give rise to a natural “distance” on S^1 (and hence to a measure of the “width opened between rays”);

(II) the map from angles to coordinates of the corresponding point in S^1 gives rise to two well-defined real functions (“cosine”, “sine”) with the usual properties.

Of course, a variety of formal and informal approaches to these questions exist, especially if we ask *only* for definitions of sine or *only* for an explanation of what angles mean; a quick web search will convince the reader that both themes are of great public interest. What we focus on here are coherent contributions to (I) and (II) which could be developed in undergraduate analysis courses.

Complex analysis approach

Rudin (1986, p. 1) begins his seminal book "Real and Complex Analysis" as follows:

> This is the most important function in mathematics. It is defined, for every complex number, by the formula
>
> $$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \qquad (1)$$

With the notations $e = \exp(1)$ and $e^z = \exp(z)$, he *defines* $\cos(t) = \mathrm{Re}(e^{it})$ and $\sin(t) = \mathrm{Im}\left(e^{it}\right)$ for $t \in \mathbb{R}$, and proves the usual functional properties based on (1). The geometric interpretation of these functions, shown in Figure 2, follows from the Cartesian representation of complex numbers. However, the meaning of "angles" is not clarified. Some ideas can be extracted from statements proved after the above definition, including the fact (ibid., p. 2) that "The mapping $t \mapsto e^{it}$ maps the real axis onto the unit circle". Combined with other elements of the exposition, including a definition of π, it is in then easy (but not done) to prove a converse: for every point z on the unit circle, there exists a unique number $a \in [0,2\pi[$ such that $z = e^{ia}$. This enables a possible definition of the angle $A(z)$ corresponding to the point z (in fact, the angle between the ray from 0 to z, and the ray from 0 to 1). We note that in complex analysis, $A(z)$ is usually called an argument of z. However, this does not really answer question (I) above: why do arguments provide a kind of *distance* on the circle? To do so, we would need to relate the above definitions to a rigorous theory of curve length in the complex plane. In particular, a theorem (not a definition) of curve length would then give us that when $0 \leq s < t < 2\pi$, the length of the arc from e^{is} to e^{it} is $\int_s^t \left|ie^{ix}\right| dx = t - s = A\left(e^{it}\right) - A(e^{is})$. And then, finally, we also have an answer to (II), as $\cos(a) = \mathrm{Re}(e^{ia})$ and $\sin(a) = \mathrm{Im}(e^{ia})$ for $a \in [0,2\pi[$.

The topological notion of curve length in the complex plane is essentially the same as that of curve length in $\mathbb{R}^2$ (as covered in vector analysis, cf. the next subsection). The above approach requires, *in addition*, a solid background on series and their convergence properties. It has the merit of relating exponential and trigonometric functions from the outset, including the characteristic differential identities. So, assuming that all prerequisites are in place, it certainly holds potential for a "Plan B" approach to (I) and (II).

Real analysis approach

A more minimal approach to question (I) and (II) can be found in the textbook (Eilers, Hansen, & Madsen, 2015), written for a second semester course on real and vector analysis at the University of Copenhagen. The textbook includes an appendix, including one section on "Trigonometric functions", which builds on chapter 7 of the book. The appendix refers to that chapter as basis for stating that there is an "angle mapping" $\gamma: \mathbb{R} \to S^1 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ which is *natural* in a sense explained in the next paragraph; this map is then used to define cosine and sine (as the first and second coordinate function of γ), and to prove the most important identities and inequalities involving those functions, including inequalities which can be used to find the derivatives (this, however, is not even mentioned as an exercise).

The meaning of *natural* only appears from a sequence of definitions and theorems in the main text (ibid., pp. 223-227, all quotes translated from Danish by the author). First some notation is introduced: for a continuous curve $\mathbf{r} : [a,b] \to \mathbb{R}^m$ and a partition D of $[a,b]$ consisting of points $a = t_0 < t_1 < \cdots < t_k = b$, we put

$$\ell(D) = \sum_{j=1}^{k} \left\| \mathbf{r}(t_j) - \mathbf{r}(t_{j-1}) \right\|.$$

A figure illustrates how this measures the total length of line segments between points on the curve corresponding to the partition, and how this can be interpreted as a lower bound of what is intuitively the “length” of the curve. Then follows the precise definition of curve length, ultimately in terms of the usual distance in $\mathbb{R}^m$:

> **Definition 7.14.** For a continuous curve $\mathbf{r} : [a,b] \to \mathbb{R}^m$ with bounded, closed parameter interval, the curve (or arc) length is given as
>
> $$\ell = \sup\{\ell(D) \mid D \text{ is a finite partition of } [a,b]\}$$
>
> If $\ell < \infty$, the curve is said to be *rectifiable* (ibid., p. 223).

It is then proved that if $\mathbf{r}$ is piecewise C^1, then it is rectifiable, with $\ell = \int_a^b \|\mathbf{r}'(t)\| \, dt$. Under the further assumption that $\mathbf{r}$ is smooth (i.e. $\mathbf{r}'(t) \neq \mathbf{0}$ for all t), it is proved that there is an interval $[c,d]$ and a strictly increasing C^1-function $\varphi: [c,d] \to [a,b]$ such that $\tilde{\mathbf{r}} = \mathbf{r} \circ \varphi$ is a *natural* parametrization defined on $[c,d]$, meaning that $\int_u^v \|\tilde{\mathbf{r}}'(t)\| \, dt = v - u$ whenever $c \leq u \leq v \leq d$. In particular, $d - c = \ell$ where ℓ is the curve length of $\mathbf{r}$. In words: there is a reparametrization of $\mathbf{r}$, such that the length of any curve segment is the distance between the corresponding parameter values.

The text gives three examples, including that after piecing together some smooth C^1-parametrization $\mathbf{r}$ of S^1, traversing S^1 once from (1,0) to (1,0) in the usual direction, we may use the above to construct a *natural* reparametrization γ of $\mathbf{r}$. It is not made explicit that $\mathbf{r}$ has finite length, and one can then *define* π as half of that length.

The appendix does not explain why γ is called an *angle map*, for instance, how it may be used to define the angle between two points on S^1. The example in the main text is likely to strike the students as trivial, even if the above “example” concludes by postulating that $\mathbf{r}$ can be used to define cosine and sine, referring to the appendix.

We observed (and co-developed) a course based on this textbook in the spring of 2015. Indeed, the overall emphasis of the course followed a Plan A, laying out theoretical foundations for real and vector analysis, with a strong emphasis definitions, theorems and proofs. The appendix was not covered by the lectures or in the exam requirements, and was only briefly used for an exercise in the first week. The above material from Chapter 7 (and more) was covered in one lecture during the sixth week. No exercises set for students addressed the problem of defining angles; the pieces are not brought together. So, even if the book and its material holds potential for a complementary Plan B related to (I) and (II), it thus seems likely that students will not realize that the theorems on curve length can be used to solve two related mysteries of their past teaching. We now turn to investigate this further.

A STUDENT'S DEVELOPMENT OF PLAN B, BASED ON PLAN A

To gauge whether the second approach could after all be noticed or at least excavated for some students having taken the course considered above, I interviewed a master level student who had served as a teaching assistant in this course twice, and is within the top 10% tier in the mathematics programme at Copenhagen. This student's knowledge would likely be an upper bound of what younger students would retain or be able to reconstruct from the course, as regards the questions of angles and trigonometry. The interview was semi-structured, based the following questions:

- What is your favourite definition of the function sine?
    - Follow-up questions according to the definition chosen, leading to:
- What is your favourite definition of angles? How do they relate to sine?
    - Follow-up questions for instance on arc length, if referring to circle arc
- What mathematical resources does the course (described above) provide to elucidate the previous questions? (textbook at hand to look up points)

The advanced student (AS) begins answering the first question by tracing the graph of sine (Fig. 4), then enumerates "a lot of properties" of the function (Fig. 5). AS realises that it is "certainly not a definition". I insist on AS giving one. AS then gives the Cartesian description (Fig. 6). After a slight confusion, AS identifies x (in Fig. 6) as the appropriate angle in the triangle.

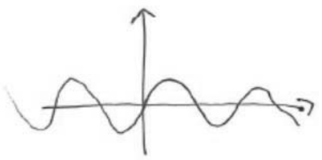

Figure 4: A graph.

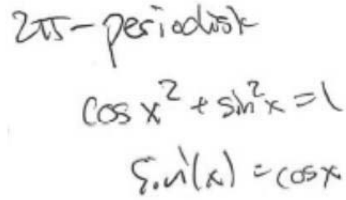

Figure 5: Properties.

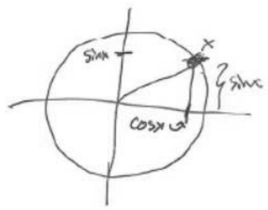

Figure 6: Diagram.

The following dialogue follows (some redundancies are left out, marked by //):

CW: Then, the question arises, the angle, what mathematical object is that? // Can you give like a mathematical definition of that?

AS: It determines like how far two lines are from each other. // [AS draws two crossing lines and says there are four angles, two different, and I repeat the previous question]

AS: In the old days, you used your compass and your protractor, and then later when you have to compute them, there were, you had these smart things where you get hold of cosine and sine to compute them, the angles…

CW: Yes, but you have just used angles to define sine and cosine…

AS: Oh yes, precisely, then it comes backwards again, so that it not so good…

CW: So, my thousand dollar question, could An0 [*the course*] help us with that?

AS soon recalls this "fantastic exercise" where you had two functions, and it turns out they are cosine and sine. After a few minutes, AS finds it in the section on curve integrals; the exercise begins with "a natural parametrization" $\mathbf{r}(t) = (C(t), S(t))$ of the unit circle, and asks the students to prove that $(C'(t), S'(t)) = \pm(S(t), -C(t))$. However, AS does not notice, at first, the assumption of **r** being *natural*, and when I point it out and ask what it means, AS looks it up in the index of the book. This leads to the definition mentioned above. AS reads the definition for a while, then says:

AS: the parameter values, they should, if we subtract them from each other // the curve length should be like one minus the other // the curve cannot make like strange crossings, that must create a mess, I think…

CW: // you get such a length preserving map from an interval onto the curve. How could that help us with the question about sine, cosine and angles?

AS: That's a good question.

I point to the appendix on trigonometric functions. AS recalls that they did do an exercise from there but "otherwise we did not look at it". AS does not recall the part on the "angle map" γ and has no idea where γ comes from. I point out the reference to the main text and the explicit mention that γ is a natural parametrization. AS returns to the main text and looks at Definition 7.14. Paging a bit, AS finds the theorem on the existence of natural parametrizations of smooth C^1-curves. I ask if AS could verify the conditions for the circle. Supported by a hint ("could you parametrize the circle without using sine and cosine?"), AS comes up with the parametrization $(\sqrt{1-t^2}, t)$, for the circle in the first quadrant. AS says this can be differentiated many times so it is C^1. AS does not recall the definition of "smooth", but quickly looks it up, and then verifies it for the parametrization above. Going back to the theorem, AS concludes that then we can construct the angle map.

CW: Using that, can you give a precise definition of what an angle is?

AS: Not immediately…

CW: What should it be, if you look at the unit circle? //

AS: It has something to do with the arc length //

AS tries to find out what the arc length is for γ and writes down the formula $\int_0^x \|\gamma'(t)\| dt$. As seen above, the meaning of "natural curve" is not familiar to AS. It also seems to confuse AS that the values of γ are clearly not angles (but points), in spite of the name "angle map". After some neutral circling around these matters, I ask AS if we could make "a function from points on the circle to arc length". AS suggests that $\gamma(t)$ should be mapped to t in some sense. As this is very close to a satisfactory answer and as the agreed time is almost up, I briefly show how to formalize this last point, and wrap up the conversation.

The 45 minute conversation outlined above suggests that the angle question would be challenging for average students and require more support than the course offers.

CONCLUSION

The formal definition (7.14 quoted above) of curve length in real analysis is a rigorous answer to questioning the intuitive idea that “nice, bounded curves have length”, relating that idea to the only case in which a proper, non-analytic definition can be made, namely for the length of line segments. When introduced to “radians”, high school students do encounter a link between curve (circle) length and angles; but the meaning of curve length is never questioned (in the sense of Chevallard, 2012) and the link to angles may soon be forgotten. The organization of the textbook (Eilers et al., 2015) and the corresponding course does not really highlight this link, and the experiment with AS suggests that the approach outlined in this paper is too ambitious for the course. Identifying viable “local Plan B’s” could contribute to diminish the “gap” which students otherwise experience between rigorous analysis and their previous knowledge (from calculus and other areas) - not to speak of their future practice, for instance if they become high school teachers.

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