

# **Should university students know about formal logic: an example of nonroutine problem**

Sarah Mathieu-Soucy

Concordia University, sarah.msoucy@gmail.com

*The goal of the study presented in this paper is to discuss how knowledge of formal logic changes the way students produce and validate proofs in the context of undergraduate mathematics. With that in mind, we asked 8 students with varied levels of knowledge and different academic background in formal logic to produce and validate proofs through a task based interview and we analyzed their work. In particular, a nonroutine task was proposed and showed interesting work from the students. Our empirical results suggest that a course in logic changes the way students do mathematical work in many ways. For example, it creates alertness to logical characteristics and a need to rely on the context.*

*Keywords: formal logic, university mathematics, undergraduate students, nonroutine problem.*

## **CONTEXT**

This paper reports on a project as part of a master's thesis (Mathieu-Soucy, 2015) where the practical role and the contribution of formal logic in mathematics were investigated. In the literature, this role and contribution is not clear. Some, for example Poincaré (1905), consider that logic is essential to mathematics and others, for example Dieudonné (1987), consider that logic is not useful to mathematics. Mathematicians Thurston (1994) and Thom (1967) claim that their basic (intuitive and theoretical) knowledge of logic is sufficient for their work and that they use different techniques instead that come, at least in part, from their experience doing mathematics. When it comes to university mathematics students, who don't have as much experience, where do they get the knowledge necessary to do mathematics without making any logical error? Selden & Selden (1999) noted that concepts studied in most beginner courses in formal logic, like Venn diagrams or truth tables, aren't that useful in the everyday mathematics students have to perform. Also, complex logical statements can often be written in multiple simple statements so that the person manipulating them doesn't need to control all the more complex aspects of formal logic. In the same line of thought, Cheng & al. (1986) showed that a course in logic does not prevent students from making logical mistakes when doing mathematics. However, among students, gaps in knowledge of formal logic are one of the causes of difficulties in validating and producing proofs (Epp, 2003; Selden & Selden, 1995). In sum, assessing the usefulness and the necessity of logic in the production and validation of proofs is quite difficult. Hence, it appears worthwhile to address this question: how does knowledge of formal logic, or a course in formal logic, changes the way undergraduate mathematics students produce and validate

proofs? This question will be addressed considering the concepts and characteristics presented in the conceptual framework below.

## **CONCEPTUAL FRAMEWORK**

To approach this question, we examine different aspects of mathematics that could help us characterize mathematical work, proofs in our case. First, we usually agree that in order to do mathematics, we need to combine intuition and rigour (which includes logic). But what is intuition? In our work, intuition is a feeling that imposes itself to an individual without being able to explain why. This knowledge arises subjectively to an individual as being true (Fischbein, 1982, 1987). Also, it comes from the experiences of each individual and it can be mathematically incorrect. Finally, regarding the use of logic in mathematical work, we recognize that logical considerations are absent or nearly so from the discourse of educators and textbooks at the beginning of university and consequently from the work of students (Durand-Guerrier & Arsac, 2003). Such considerations are replaced by contextualized reasoning rules and contextualized knowledge, specific to a certain field of mathematics. Their use seems to be directed by the mathematical knowledge of the individual or his mathematical expertise.

## **METHODOLOGY**

Considering that there could be contextualized knowledge as a result of a course in logic, that a course in logic is not the only source of knowledge in logic (mathematical experience, for example) and with the possible characterizations mentioned above, we developed a methodology in two phases involving eight university students from Quebec, Canada in the second half of a 3-year mathematics program (20-21 years old). First, we evaluated their level of knowledge in formal logic with a written test. Questions in this setting were strictly formal. There was no proof to be done in this test, only direct questions on logical ideas (finding the negation of a statement or defining a *modus ponens*). Then, considering those results (ranging from 0 to 4), and their academic background in logic (if they did or did not take a course in logic), we formed 4 different teams of 2 students to move to the second phase (see below). The four teams were as follows: Anna and Michel formed the only heterogeneous team, meaning Anna did not take a course in logic and was the weakest student on the test (0/4) while Michel did take a course in logic and he was one of the strongest on the test (3/4). The second team, Jeanne and Lucie, were considered as having the same profile as Michel (3/4 and a course in logic). The third team consisted of Éléanore and Paul, who did not take a course in logic and showed a slightly less knowledge in logic in the test (2/4). Finally, Julie-Ann and Robert formed the fourth team. They were the students who showed the most knowledge in formal logic in the test (4/4) and they did a course in logic. We should note that the five students who have academic background in logic did the exact same course at the same time.

The second phase of the methodology consisted of audio-recorded task based interviews (Goldin, 1997). We asked each of the four teams to answer four questions (see Appendix 1 for Questions 1, 2 and 4, Question 3 below). They needed to reach a consensus on the solution at the end of the resolution. The task that this paper is mostly interested in is the third one, which is our adaptation of the first three Hilbert's incidence axioms (Arsac, 1996). This was the only task with an unfamiliar context. While still answering to regular mathematical rules, this question does contain an additional difficulty for the students, other than the logical difficulties. Indeed, conceptualizing new objects (dogs, robots and a friendship relationship) in a known mathematical framework is rarely done in a student's life.

**Question 3**

*Here is a question asked to a university student :*

With the following axioms :

- *AX1* : For every robot  $L$  and for every robot  $J$  different from  $L$ , there exists an only dog  $W$  which is friend with  $L$  and with  $J$ .
- *AX2* : For every dog  $W$ , there exists at least two distincts robots which are friends with  $W$ .
- *AX3* : There exists three distincts robots such as no dog is friend with those three robots simultaneously.

Show that for every robot, there exists at least one dog which is not his friend.

*Here is his answer :*

Let's consider a generic robot  $R$ .

Let's consider  $L1$ ,  $L2$  and  $L3$  being the three distinct robots such as no dog is friend with those three robots simultaneously (*AX3*).

If  $R$  is the same robot as  $L_i$ , for  $i=1$  or  $2$  or  $3$ , the dog is easy to find : we take the only dog which is friend with the two other robots, its existence is guaranteed by *AX1*. This dog cannot be friend with  $L_i$  without contradicting *AX3*.

We can then suppose that  $R$  is different from  $L1$ ,  $L2$  and  $L3$ .

**Complete the proof.**

**Figure 1: Translation from French of Question 3**

The three other tasks involved familiar contexts, on material from first-year university courses. A posteriori, questions 1, 2 and 4 were considered too easy to get any useful data to give an insightful answer to our question in a familiar context. However, the students work on those tasks was still useful as a basis of comparison, and per respect to other aspects mentioned in the results section.

## **RESULTS**

Our analysis of the participants' work suggests that a course in logic changes the way students produce and validate proofs. Our results were, however, inconclusive regarding any difference between students with significantly different results on the pre-test on formal logic.

### **Alertness and Uneasiness**

An academic background in logic seems to: increase the alertness to logical characteristics, promote an unconscious notice of logical specifications for students and increase their ability to “unpack” (Selden & Selden, 1995) logical characteristics of symbolic and discursive statements. We hypothesize that it is due to intuitions and contextualized knowledge resulting from a course in logic.

To be more precise, this alertness seemed to be very useful as the students who did a course in logic noticed quickly the logical considerations contained in all four questions, as if it was jumping out of the page. Indeed, they were aware of every logical detail of the questions very quickly and could reflect accordingly from the beginning, while others might reread the same statements many times and still miss some implications or quantification and lose time reflecting on a slightly different problem.

On the other hand, students who showed alertness also showed uneasiness to engage into the mathematical work, especially in Question 3. Indeed, teams formed by two students who showed increased alertness took a significant amount of time to solve the question compared to the other teams (48 minutes for Jeanne and Lucie, 38 minutes for Julie-Ann and Robert, compared to 19 minutes for Anna and Michel and 22 minutes for Éléanore and Paul). It was surprising to us at first that students who did the best on the test and the ones that did the course in logic struggled the most and took the most time. Indeed, we expected that the necessary knowledge for the nonroutine task was closer to pure logic than for the other questions. When looking closely at the sessions, the extra time results from moments where students discussed the problem without engaging in the process of producing a proof. For example, they discussed the axioms and their role (by giving examples of settings with different relationships) or how to address the problem (by contradiction or by using the second axiom first).

The uneasiness was particularly flagrant when looking at the heterogeneous team, formed by Anna, who did not do a course in logic, and Michel, who did. Indeed, Michel was the leader of his team when resolving questions 1, 2 and 4. He was much quicker than Anna in every way. However, in Question 3, the opposite happened. Michel, as the other students who did a course in logic, was hesitant to enter the task while Anna jumped right in, trying many different approaches until she found the right one. Michel was just along for the ride. He was hesitant to try any idea he

would get while Anna was not. In both cases, we do not think that Anna or Michel lacked knowledge to solve any of the tasks.

We think that multiple things can cause the uneasiness. For example, the large amount of logical considerations present in this task could be a cause. Indeed, with their increased alertness, the students could have been blinded by the logical structure in a way that slows their progress in the task. Also, since they are more aware of the intricate logical structure, they are aware that the risk of error might very well be greater than usual and think they should refrain from going forward in the resolution until they have a better control over the problem, over the context.

### **Control Over the Context**

When solving the dog and robots' task, the same students who did a course in logic seemed to seek the control they normally have over the context (*what makes sense in a particular situation*). A great example of this idea is when Jeanne became frustrated because Lucie and herself struggled to finish the proof:

Jeanne: Why can't I find the solution?

Lucie: It's only because it's about robots and dogs.

Here we see that for Lucie, it was obvious that the absence of semantic ground on which to rely on as they were solving this task was the root of the whole problem, as opposed to the intricate logical structure or any other characteristic of the problem. It is indeed true that having a control over the context can help in a resolution. For example, let's look at the axioms in their original form: the first three incidence axioms of Hilbert, as formulated by Arsac (1996). It would become:

- Through 2 distinct given points, one and only one line is incident to both
- Given a line, there exists at least two distinct points incident to this line
- There exist three non-collinear points<sup>1</sup>

In this case, students would probably be able to assess the righteousness of their reasoning relying on the context. For example, if at some point in the resolution, the students would infer the following statement: *Given a line, there exists a unique point incident to this line*. It would be easy for the students to discard this statement about points and lines since it is obvious that there is more than one point incident to a line according to our knowledge of points and line. However, considering the equivalent statement *Given a dog, there exists a unique robot who is friends with this dog*, it is not shocking that a dog would have an only robot friend! Hence, we hypothesize that many students were resistant to engage into the task in the

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<sup>1</sup> My translation of: Par deux points distincts donnés, il passe une droite et une seule; Étant donnée une droite, il existe au moins deux points distincts sur cette droite; Il existe trois points non alignés.

beginning, until they were “familiar” enough with the relationship between dogs and robots to be more confident. For example, Julie-Ann and Robert discussed the three axioms a lot before doing a step toward the proof they were asked to produce, giving examples of what kind of friendship could exist. At this occasion, Robert questioned the second axiom extensively to finally exclaim himself:

Robert : Oh! I was wondering what is the use of AX2: there is no useless dog.

At that point, Robert was more comfortable with the task because he knew that no dog was friendless and it gave him insight on this new “world” populated by robots and dogs. Also, since the second axiom was not used in the beginning of the proof given to the students (see figure 1), it was even more important for them to find its role, what it really meant in the context.

## Symbols

Some students transformed the axioms into symbols to increase their understanding and control over the logic involved. For example, in some of Jeanne and Lucie’s work (see figure 2), they let  $\sim$  be the symbol for friendship,  $R_0$  the set of all robots and  $Ch$  the set of all dogs, they symbolized the axioms:

- AX1 :  $\forall L, J \in R_0$  such as  $L \neq J, \exists! W \in Ch$  such as  $L \sim W$  et  $J \sim W$ .
- AX2 :  $\forall W \in Ch, \exists L, J \in R_0$  such as  $L \neq J$  et  $L \sim W$  et  $J \sim W$ .
- AX3 :  $\exists L_i \neq L_j \neq L_k \in R_0$  such as  $\forall W \in Ch, W \sim L_i$  et  $W \sim L_j \Rightarrow W \nmid L_k$ .

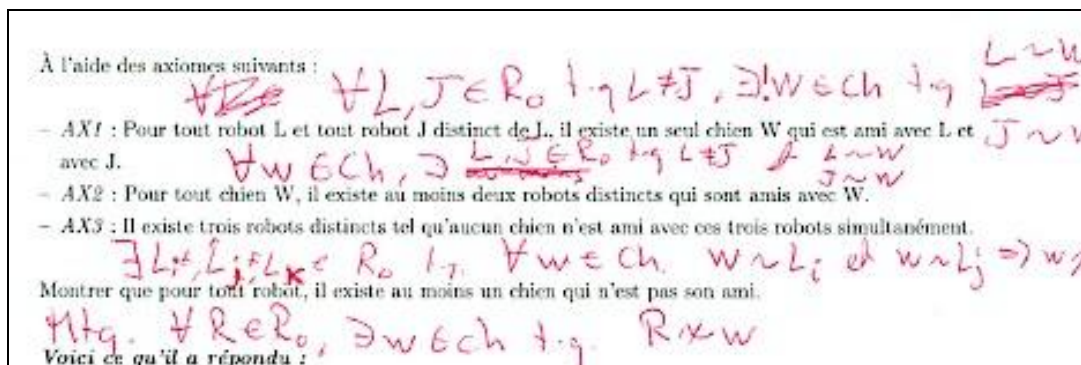


Figure 2: example of the symbolization of the three axioms presented above

The symbolization proposed by this team involved all the right logical specifications (while it is not the most rigorous, we still consider it conveys the right ideas). Jeanne and Lucie mentioned explicitly that this symbolization was necessary for them to grasp the meaning of the axioms. However, later on in the resolution, both Jeanne and Lucie mentioned that they were only looking at the discursive statement and not at the symbolize version to remind themselves and reflect on the axioms during the actual proving process. The symbols helped them grasp the ideas underlying each statement with a more synthesized and compact version, but the symbols became useless for them in the middle of the proof process, as if the symbols were opaque

compared to the words. Also, if reading the discursive version of the axiom did not help them, Jeanne and Lucie would symbolize the axiom they were interested in again from the beginning, on another piece of paper. This shows how much the work of transforming into symbols, and not only the symbols themselves, is helpful to grasp and convey ideas in mathematics.

### **Does alertness and relying on the context prevent making mistakes?**

It would be a mistake to think that being alert (as a result of doing a course in logic) and grasping the meaning of the axioms in the context necessarily means understanding concepts and being able to use them properly. For example, Jeanne and Lucie, who put into symbols the axioms of the nonroutine problem as shows figure 2, considered every logical specification when rewriting the axioms but they thought there was a one-to-one correspondence between  $AX1$  and  $AX2$ . We can easily see that it is not the case since  $AX1$  specifies the existence of a unique dog while  $AX2$  specifies that there exists at least one robot, eventually many. Hence, even if they saw the difference in the quantification and knew what the quantification meant, they still made a mistake and were not able to resolve the task properly. Similarly, the students who did miss some logical specification in the statements (those students coincide in our study with students who did not do a course in logic), and were consequently making mistakes by omitting information, often showed in the test prior to the interview that they had the formal knowledge associated. For example, after working for a couple minutes to solve the task, Paul said: “ ‘there exists an only’. I read it, I wrote it  $[\exists!]$  but I did not take it into consideration. This is why it took us time to conclude.” In this case, Paul read many times the axioms and reflected on them without “absorbing” the special quantification associated with the dog in  $AX1$ . While it took his team longer to conclude because of this omission, he did not make any mistake and knew exactly what it meant and how to use it, once he actually noticed.

### **The influence of the type of context (formal, familiar, unfamiliar)**

Our data suggests that the type of context influences the students’ ability to perform manipulations associated with formal logic. Every student was confronted with three different contexts during the experiment. In the written test, we consider the context as being strictly formal. During the task based interview, there were three tasks that presented a familiar context: the questions involved mathematical objects and concepts that the students saw many times in many classes. Finally, the third task involved an unfamiliar context: the object and relations were *a priori* unknown by the students but still answered to the same general mathematical rules.

Six of the eight students struggled to find the negation of “ $A \Rightarrow B$ ” (in the test prior to the interview) but they were all perfectly able to negate “the double of any irrational number is irrational” (Question 1). Hence, it seemed important for them to reflect in a familiar context, namely what does it mean for the statement to be false,

according to their knowledge of the concept. Similarly, Julie-Ann struggled to negate “For every robot, there exists at least one dog which is not his friend”, but was perfectly able to negate “ $\forall f \exists a [(\forall u(F(u,a) \Rightarrow G(f,u,a)) \Rightarrow H(f,a)]$ ” (Durand-Guerrier & Njomgang Ngansop, 2009). So for her, negating in a formal context was easier than negating in an unfamiliar context. In other words, control over formal statements is not a necessary and sufficient condition to the control over statements in context and, thereby, on their semantics.

## CONCLUSION

Our results suggest that a course in logic did change the way student worked, especially in the case of a nonroutine task. It increased their alertness to logical characteristics while creating uneasiness to progress in a resolution. In both cases of vigilant and less vigilant students, their vigilance was the same when confronted with logical characteristic they could work on and understand or not. This implies what Cheng & al. (1986) already mentioned, namely that a course in logic does not eliminate the risk or errors when working on logical considerations in a mathematical context and also that the production and validation of proof is not necessarily improved by such a class. Also, the students less vigilant would benefit from looking for the logical characteristics more explicitly.

Going back to the title: should university students know about formal logic? While answering this question was not the prior goal of the master’s thesis associated to this work, the results still bring some pieces of answer to this question. This paper shows once again that extensive knowledge of formal logic is not necessary to do mathematics. However, what this research brings is the whole idea of alertness to logical characteristics, which is an interesting asset for mathematics students. What this research also reminds us is that noticing is useless without a strong hold on the notions. This brings us to expand our reflection to the teaching of logic: what kind of knowledge should be taught and in what way to promote students’ understanding and diminish logical mistakes, in order to make logic courses as efficient as possible?

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## APPENDIX 1

Translation from French of the tasks proposed in the task based interview

### Question 1

*A university student suggested this proof. Address a comment to him.*

Theorem : The double of any irrational number is irrational

Proof : Suppose it is not. That is, suppose the double of every irrational number is rational. But we previously proved that  $\sqrt{2}$  is irrational and also that  $2\sqrt{2}$  is irrational. These results contradict our supposition. Hence the theorem is true.

(Epp, 1997)

### Question 2

Study the following conjecture : *The composition of two surjective functions is surjective.*

(Epp, 1999)

### Question 4

*A university student suggested this proof. Address a comment to him.*

Proposition : Let's consider  $E$  and  $F$  two sets and  $f$  an application from  $E$  to  $F$ . Whatever parts  $A$  and  $B$  from  $E$ , we have  $f(A \cap B) = f(A) \cap f(B)$ .

Proof :

Lets prove first that  $f(A \cap B) \subset f(A) \cap f(B)$  : if  $f(x) \in f(A \cap B)$ , then  $x \in A \cap B$ ; since  $x \in A$ ,  $f(x) \in f(A)$ ; and since  $x \in B$ ,  $f(x) \in f(B)$  and then  $f(x) \in f(A) \cap f(B)$ .

Let's now prove that  $f(A) \cap f(B) \subset f(A \cap B)$  : if  $f(x) \in f(A) \cap f(B)$ ,  $f(x) \in f(A)$  then  $x \in A$ ; also  $f(x) \in f(B)$  then  $x \in B$ ; since  $x \in A$  and  $x \in B$ ,  $x \in A \cap B$  and then  $f(x) \in f(A \cap B)$ .

(Durand-Guerrier, Barrier, Chellougui & Kouki, 2012)