Envelopes in a computerized environment:
the transition from 2D to 3D

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Abstract: The usage of a Computer Algebra System (CAS) allows new approaches to classical topics. We describe how envelopes of parameterized families of curves and surfaces can be studied using technology for automatic solutions of equations, dynamical representations and experimentation. Work is based on the knowledge acquired in Calculus and Advanced Calculus. The study of envelopes enables also to make connections with other mathematical domains, such as Abstract Algebra in order to have access to other algorithms implemented in CAS. The transition from 2D to 3D is non-trivial, as technological features available for 2D may either not exist or be harder to use in a 3D setting.

Keywords: Differential Geometry; Envelopes; Computer Algebra Systems; dynamical representations.

INTRODUCTION

The intertwining of experimental approaches and theoretical thinking is the core of discovery in every mathematical field. The usage of Computer Algebra Systems (CAS), packages for Dynamical Geometry Systems (DGS) and other technologies add important tools to the experimental part of the work. Arzarello et al. (2012) mention that the latest developments in these technologies have "rejuvenated mathematics and mathematics education", as they provide numerous new computational and symbolic tools, and revitalize experimental mathematics and visual tools.

Classical topics in Differential Geometry like the study of 1-parameter families of curves in 2D space and of surfaces in 3D space and their envelopes have been abandoned in the past for various reasons (Thom, 1962). We may mention only the fact that the theory has few powerful theorems and numerous special cases. Visualization is also a crucial issue, in particular in 3D. Moreover, the computations may be quite heavy and unilluminating.

Nevertheless, this topic has a great interest in mathematics and in applied science. Envelopes have numerous applications in science and engineering: caustics and wave fronts (geometrical optics, theory of singularities), robotics (rigid motion in 2-space and in 3-space, collision avoidance), diffusion of chemicals in the soil, etc.

In this paper, we show how the envelopes can be revived and studied making a joint usage of paper-and-pencil together with both CAS and DGS. This is an approach recommended in (Roulier & Böhm, 2015) and (Wegscheider, Himmelbauer, & Böhm, 2015). The examples are based on experimental and computational work.

The transition from 1-parameter families of plane curves to 1-parameter families of surfaces in 3D space presents a double trend: the algebraic part follows a similar pathway in both settings, but visualization is different. The algebraic components of CAS are well developed, and actually the same algorithms are at work. The availability in the software of dynamical features, in particular a slider bar, as in GeoGebra, is a core issue for building intuition and understanding in 2D: tracing changes yields a construction of the desired objects (see Section 2). Direct manipulation of the figures using the mouse is still rare in 3D software.

The topic has been taught during last years in sessions for both pre-service and in-service teachers.

ENVELOPES OF 1-PARAMETER FAMILIES OF PLANE CURVES

Let a family of plane curves be given by the equation $f(x, y, t) = 0$, where t is a real parameter. An envelope of the family, if it exists, is a curve tangent to every curve in the family (Struik, 1950). It can be shown that this envelope is the solution set of the following system of equations (op. cit., p. 167):

(1)
$$\begin{cases} f(x, y, t) = 0 \\ \dfrac{\partial f}{\partial t}(x, y, t) = 0 \end{cases}.$$

Figure 1a shows the envelope of the family of lines given by the equation $x + ty = t^2$ (it is the parabola whose equation is $x = -y^2/4$). Figure 1b shows the envelope of the family of unit circles centred on the parabola whose equation is $x = y^2$.

(a) A family of lines   (b) A family of circles centred on a parabola
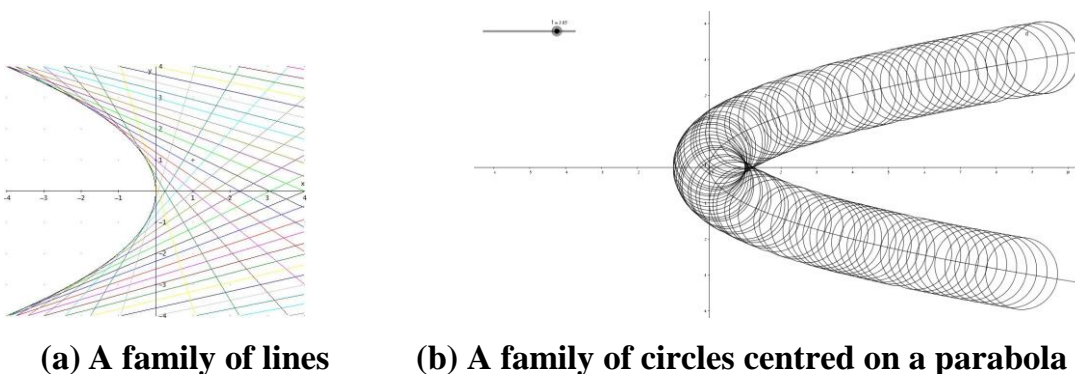
Figure 1: Two examples of envelopes

The usage of a slider bar enables to build envelopes experimentally. The experimental study may enhance the understanding of various properties, among them:

a. Possibility of non-existence of an envelope.

b. Possibility of non-unicity of an envelope.

c. If an envelope exists, the existence of singular points.

The envelope of the family of lines in Figure 1a has one component and no singular point. In Figure 1b, the envelope of the family of circles has two components; the exterior one seems to be smooth, the interior one seems to have singular points, two cusps and one double point. This last property needs to be checked; this may be done using the algebraic presentation of the technology.

A general point on the given parabola has coordinates (t^2,t), whence a general equation for the circles: $(x-t^2)^2 +(y-t)^2 -1=0$. Denote by $f(x,y,t)$ the left-hand side of this equation. Equations (1) defining the possible envelope of the family read here, after simplification:

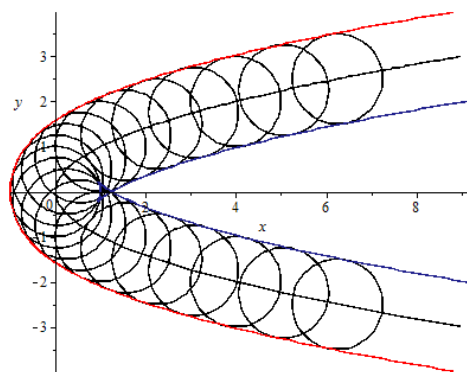$$\begin{cases} x^2 - 2xt^2 + y^2 - 2ty + t^4 + t^2 - 1 = 0 \\ 2tx + y - 2t^3 + t = 0 \end{cases}.$$

Using the **solve** command of the CAS, we obtain:

$$(x, y)=\left(t^2 + \frac{1}{\sqrt{4t^2 +1}}, \frac{-2t}{\sqrt{4t^2 +1}} +t \right) \text{ or } (x, y)=\left(t^2 - \frac{1}{\sqrt{4t^2 +1}}, \frac{2t}{\sqrt{4t^2 +1}} +t \right),$$

showing the existence of two components. Figure 2a shows the two components and Figure 2b shows the parabola, a few circles and the envelope. These figures have been obtained with parametric plot.
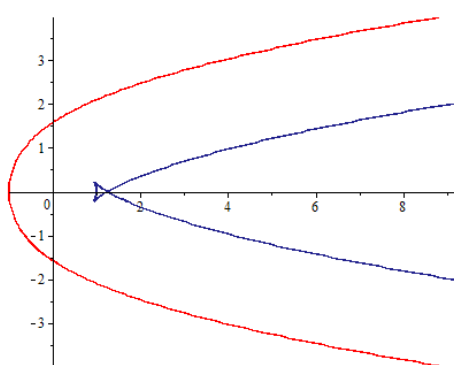
(a)   (b)

Figure 2: Envelope of a family of unit circles centred on a parabola

Elimination of the parameter in order to obtain an implicit equation requires the usage of a specific algebraic package of the CAS; we present this issue later.

## ENVELOPES OF 1-PARAMETER FAMILIES OF SURFACES

The transition to parametrized families of curves and surfaces in 3D space rely on the same techniques and a generalization of Equations (1). New issues have to be addressed: general visualization problems, availability of appropriate features in the software, etc. Let a family of surfaces be given by an equation of the form $F(x, y, z, t) = 0$, where t is a real parameter. If an envelope exists, it is determined by the solutions of the system of equations

(2)
$$\begin{cases} F(x, y, z, t) = 0 \\ \dfrac{\partial F}{\partial t}(x, y, z, t) = 0 \end{cases}.$$
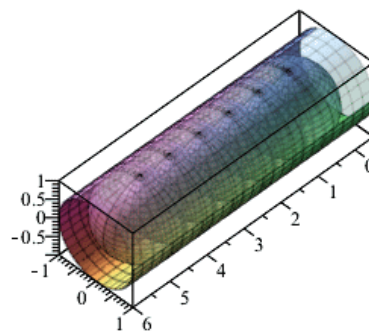
A family of spheres

Let be given the family of spheres of radius 1 centred on the x-axis. A general equation for the spheres is $(x - t)^2 + y^2 + z^2 - 1 = 0$. If an envelope exists (Figure 2), it is defined by the system of equations $\begin{cases} (x - t)^2 + y^2 + z^2 - 1 = 0 \\ 2x = 0 \end{cases}$ whence $y^2 + z^2 = 1$. This is the equation of the cylinder whose axis is the x-axis and whose basis is the unit circle centred at the origin in the yz-plane (see Figure 3b).
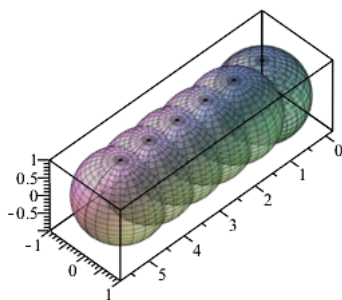
(a) The spheres   (b)The envelope

Figure 3: Envelope of a family of spheres

Intuition and visualization have been enhanced in Figure by using three steps: (a) some spheres of the family, (b) a partial plot of the cylinder and (c) a plot with the cylinder. In more complicated cases, this way of displaying may help, as transparency is not always enough to fully understand the drawing.

A family of planes

A family of planes is given by a general equation $u(t)x + v(t)y + w(t)z + 1 = 0$. In this case, Equations (1) read as follows:

$$\text{(10)} \qquad \begin{cases} u(t)x + v(t)y + w(t)z - 1 = 0 \\ u'(t)x + v'(t)y + w'(t)z = 0 \end{cases}.$$

This system has either no solution (for example when u, v, w are affine functions of the parameter t) or the solution determines a line in space. This last case means that an envelope exists and is generated by lines, i.e. the envelope is a *ruled surface*. Moreover a) this surface presents a cuspidal edge, i.e. a curve whose points are singular points, b) the tangents to this curve are generators of the envelope, c) this curve is an envelope of the family of generators (Ferreol, 2009).

Consider the family given by the equation $tx + y + t^2 z = 2t^3$ where t is a real parameter. The given family has an envelope, displayed in Figure 4 (obtained with Maple). This envelope is determined by the system of equations $\begin{cases} tx + y + t^2 z - 2t^3 = 0 \\ x + 2tz - 6t^2 = 0 \end{cases}$.

Solutions of the system are given parametrically by

$$\text{(11)} \qquad (x, y, z) = \left(-2st - 6t^2, st^2 + 4t^3, s\right).$$

Eliminating the parameter yields an implicit equation:

$$\text{(12)} \qquad x^2 z^2 - 4yz^3 - 8x^3 + 36xyz - 108y^2 = 0.$$

The cuspidal edge, called *edge of regression,* appears. Students learning Advanced Calculus know how to find singular points of a surface given parametrically (the first partial derivatives vanish at these points). The mesh chosen by the software to plot the envelope is built on the tangents to the edge of regression.

In order to understand the surface visually, the dynamics of the software are important. Generally, the software enables to rotate the drawing using the mouse, but a slider bar enabling to manipulate directly the plots and to trace the different plots is not available in 3D. In Figure 4, two still pictures have been chosen in order to enable a better understanding, in particular the edge of regression and its tangents. Note that the structure of the envelope as a ruled surface can be revealed by technology.
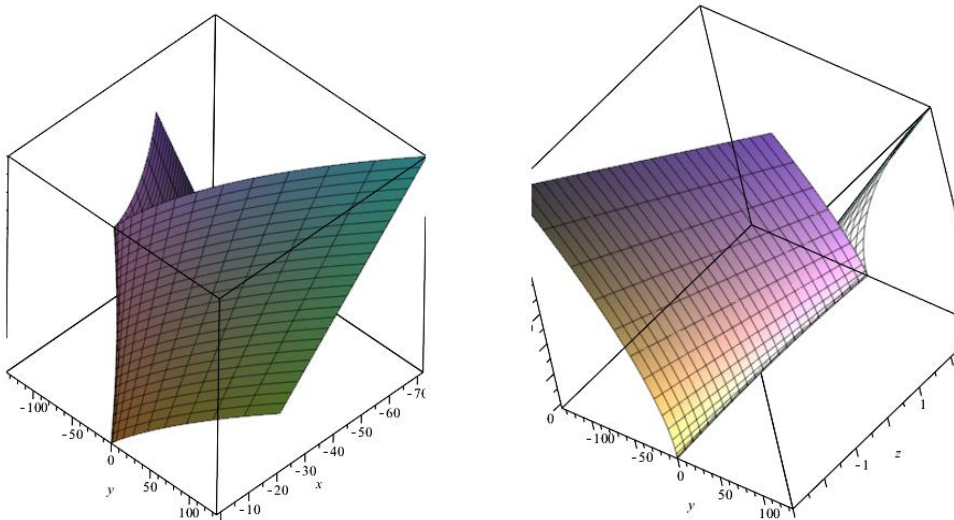
Figure 4: Envelope of a family of planes: two different views

IMPLICITIZATION - ELIMINATION OF THE PARAMETER

Both in 2D and in 3D, the solution of Equations (1) is obtained in parametric form. It happens that an implicit equation is requested. This may be done by elimination of the parameter, an operation often made possible within a specific package.

The algebraic engine to solve polynomial equations and systems of polynomial equations in different CAS is often based on computations of Gröbner bases, a domain belonging to abstract algebra (ring theory). Adams and Loustaunau (1994) explain how this may be viewed as a generalization of the Gauss algorithm for solving systems of linear equations. They provide a nice scaffolding to learn the topic, first leading hand-made computations, and then enabling to understand the algorithms implemented in software. The implementation enables to solve systems of equations, yielding a parametric representation of the solutions. Some packages provide also a tool for implicitization of this parametric representation. The Gröbner bases algorithms have been widely used by Pech (2007) for a new point of view on classical problems in geometry[1].

In the first examples presented here, the parameter appears in polynomial expressions only. Therefore the algorithms in use are based on computations of Gröbner bases. Sometimes, a non-polynomial parametrization can be replaced by a rational one, for example sine and cosine may be given by $\cos t = \dfrac{2t}{1-t^2}, \sin t = \dfrac{1+t^2}{1-t^2}$. Then multiplying by the common denominator of the expressions, polynomial

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[1] Pech uses a software specifically designed for Gröbner bases computations, the CoCoA System, freely downloadable from cocoa.dima.unige.it

equations are obtained, and from this point the Gröbner bases algorithms may be run. One of the authors taught Gröbner bases to 3rd year pre-service teachers.

If no rational parametrization is available for the family of curves (resp. surfaces), then the same **solve** command of the CAS is used to solve the system of equations defining the envelope. In this case the pattern recognition of the software leads to other algorithms; the output may also be harder to grasp.

WHAT CAN WE LEARN? REFLECTION ON OUR STUDY OF ENVELOPES

CAS literacy for understanding the algorithms

The **solve** command may hide algorithms unknown to the user and the usage of this command may look like a *black box*: the user enters data and receives output, but has no idea of what happened inside. According to the mathematical background of the students and to their CAS literacy, the teacher may consider this as an opportunity to enhance more profound understanding of the mathematics implemented into the CAS command.

Such a situation, with its pedagogical implications, has been described in another mathematical domain by Steiner and Dana-Picard (2004), namely for the computation of either definite or improper integrals. A theorem is presented, which is generally not given in textbooks and is not implemented in CAS. The theorem reads as follows: if the function f is integrable on the interval $[a,b]$ with $a \le b$, then $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$. A proof of this theorem is given by Dana-Picard (2005).

Using this theorem, the computation of the parametric integral $I_r = \int_0^{\pi/2} \dfrac{1}{1+\tan^r x}\,dx$ is an easy task. The Derive software computes the integral I_r for the general parameter *r,* not only for specific values, and shows that for any value (even a non-integer value), the integral is equal to $\dfrac{\pi}{4}$.

The fact that Derive is able to compute this integral and other packages are not, was the incentive to understand which mathematical theorem is implemented in the software. The clue is given by the step-by-step feature of Derive.

Technology may be used as a black box, but more important is to catch opportunities to have insight into the mathematics implemented in the software. In the study of envelopes, we meet a similar situation. Using rational parametrization for the curves or the surfaces in the given family, Equations (1) or (2) can be translated into systems of polynomial equations. These systems are solved by the CAS using algorithms based on computations of Gröbner bases of polynomial rings. Students having learnt a first course in Ring Theory may understand these algorithms.

If the parametrization is not rational, the solution of Equations (1) or (2) may involve more complicated expressions, even special functions. This opens a new field of exploration for an undergraduate student.

Interaction between mathematical domains

The interdisciplinary aspect of differential geometry is renewed by technology. Techniques from analysis, abstract algebra and geometry are used within a single session. The title of textbooks for the study of curves and surfaces read as "applications of analysis to geometry" (Cagnac, Ramis, & Commeau, 1966).

Because of this interdisciplinary aspect, the study of envelopes of parametrized families of curves and surfaces provides an opportunity to discover new topics beyond the scope of the regular curriculum, sometimes together with applications to practical situations. Connections between mathematical domains may be discovered and explored. New computational skills with technology may be developed, in particular for the experimental aspect of the work (conjecturing the existence of an envelope and its properties, exploring the existence of cusps, as in Figure 1b). For this, the availability of a slider bar, together with a **Trace on** option, is a central issue.

A great advantage of a CAS is its multi-purpose structure. It enables switches from one register of representation to another. Switching from the algebraic register to the graphic one is standard, but not the reversed direction. GeoGebra has the reversed switching, even when the slider bar is in use. The joint usage of a CAS and an external grapher does not provide such a switch.

There exist freely downloadable graphers[2], which enable direct manipulation of plots. These graphers are useful, but they can only be used side-by-side with a CAS and they do not collaborate and do not communicate with the CAS. Moreover, the ability to switch between different registers of representation may be improved, within mathematical domains (parametric vs implicit) and with the computer (algebraic, graphical, etc.).

Finally, Duval (2006) pointed out that there is a basic difference between mathematics and the other domains of scientific knowledge. In contrast to phenomena of astronomy, physics, chemistry, biology, etc., mathematics used not to be accessible by perception or by instruments (microscopes, telescopes, and measurement apparatus). The usage of technology has changed the situation: mathematics became accessible to experimentation.

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[2] For example, MathStudio. It can be used online at http://mathstud.io/welcome/

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