

Rationality and concept of limit

Thomas Lecorre University Joseph Fourier in Grenoble, Fourier Institute, France,
thomas_lecorre@orange.fr

We present a didactic situation aimed at the formal definition (delta-epsilon) of the limit of a function; the experimentation of this didactic situation has been made many times with French students in the last year of high school and the first year of university using "scientific debates" between students. From an excerpt of the script of an experimentation, we study the evolution of students' reasoning. We specifically study the kinds of rationalities used by students to try to solve the given problem.

Keywords: limit, rationality, definition, calculus, scientific debate.

CONTEXT

Teaching the definition of the limit of a function is known to be difficult and didactic studies related to this topic started more than thirty years ago. The large number of studies on this subject shows the efforts of the didactic community to address this difficulty which corresponds to the foundation of calculus (Robert 1982, Sierpiska 1985, Bloch 2000, Mamona-Downs 2001, Przenioslo 2004, Ghedamsi 2008,...). We will look specifically at the beginning of calculus in the last year of lycée, in Terminale S (last year of high school). The notion of limit of a sequence is introduced the year before (première S) with intuitive approaches where the dynamic conceptions of limit are given as: "the function tends to L", or in terms of a function "approaching" a number. Then, in Terminale S, a more formal definition is given. The official texts specify that "to express that a sequence u_n tends to L when n tends to + infinity, we say that any interval that contains L contains any value of u_n from a certain rank n.". However, in this article we specifically discuss the introduction of the definition of the limit in delta-epsilon of a function in the level of Terminale S (for all $\varepsilon > 0$, there exists a delta such that for all x greater than delta, $f(x)$ is between $L - \varepsilon$ and $L + \varepsilon$).

PROBLEMATIC

Our aim is to add to the first quite dynamic conceptions new conceptions called static ones, corresponding to the common formalism in calculus, which means showing this formalization as determined by two quantified systems of proximity (delta and epsilon). Studies show (Robert 1982, Williams 1991) that the dynamic conception is very resistant at university, which can be easily understood: academic formalism is not only very complex (because of quantifiers, real numbers, linked variables, the notion of infinity,...) but it can also give first the feeling that it has nothing to do with the object it is supposed to define.

We think that a part of the difficulty comes from the kind of rationality really available for students. How can a student understand such a definition if he usually does not really need any definition to use mathematical objects (Job 2011)? How can he understand the subtleties of this definition if he does not feel that this responds to a

problematics of proving calculus results, i.e. if he does not really feel that mathematics are a special building where definitions, bricks, are fit for demonstrations, as a special cement (Lakatos 1984)? How can he accept such a “repulsive” definition whereas he feels he already has a good idea and a very much easier one of what it can be, i.e. how can he accept without adopting a theoretical rationality? We finally reverse all these questions and ask: can the study of the limit be an opportunity for developing theoretical rationality? And how?

We distinguish three kinds of rationalities. Pragmatic rationality consists in closely examining specific cases. There is no attempt to generalize observations. The Empirical rationality is used when one wants to have a general law. One uses facts to deduce generalizations. Theoretical rationality begins with theory (theorems, properties, definitions, axioms...) to establish new properties and theorems (more details will be given later).

We approach this problem of rationality by creating a didactic environment in which students have no indication about the kind of rationality to use to solve the limits problems they are given; and we will follow the evolution of the kinds of rationalities used by them. We suggest a didactic engineering in two steps.

Didactic engineering is a didactic research method which is quite common in French didactic research (González-Martín & al. 2014). A specific subject is first studied in terms of theory, then a didactic experimentation is designed and an a priori analysis formulated. The experimentation is conducted and finally a comparison is made between what was expected and what really happens. The study of any eventual discrepancy between the hypothesis and the result often results in the discovery of something new and unexpected.

The aim of the first step of our didactic engineering about the limit is to destabilize the first conception of the limit (the dynamic one) and to create a need for formalization. The aspect of rationality targeted/singled out in this first step is the way that objects are given and exist in mathematics: by definitions. The aim of the second step is to give a meaning to the kind of formalization used by the definition in delta-epsilon of the limit of a function. The aspect of rationality targeted in this second part is the way to establish the truth in mathematics: by demonstration. In this article, we study this second step. We are using a "scientific debate" (Legrand 2001) where the reasoning of a group of students is not led by the teacher but is the result of interaction within the group. The corpus used in this article comes from experimentations in lycée in TS (2014 for second step). The involved students are accustomed to this type of activity.

DESCRIPTION OF THE DIDACTIC ENGINEERING

We first have a look at the first step (widely described in Lecorre 2015). The teacher gives a conjecture about two functions having no infinite limits in infinity.

C1: "If the limit of f is strictly less than the limit of g in plus infinity then for every real x , $f(x)$ is strictly less than $g(x)$ "

This conjecture is obviously false and students quickly agree that it is false. Then, the teacher asks the students to "repair" the conjecture: the main idea, the order of the limits gives the order of the functions, must be kept but the conclusion must be changed in order to make the conjecture true. The aim, of course, is to make the students confront the need to formalize the concept of neighborhood of infinity. The first suggestion given by a student is:

C2: " If the limit of f is strictly less than the limit of g in plus infinity then $f(x)$ is upper bounded by $g(x)$ in infinity".

The debate between students leads them to say that C2 looks like "if A then A" which they find pointless ("It's true without any demonstration"); then emerges the question of the definition of "f(x) is upper bounded by g(x) in infinity". (the ask for a definition shows that the students are here adopting a theoretical rationality). The teacher, then, introduces the idea of neighborhood of infinity as an answer to this question and its formalization with the conjecture.

C3: "If the limit of f is strictly less than the limit of g in plus infinity then a real A exists so that for every real x more than A , $f(x)$ is strictly less than $g(x)$ ".

The debate among students for C3 leads to a broad agreement about its validity, but some students remain uncertain considering that it has not been proven yet. The teacher then suggests to study what he calls "the monster" (see below), a function f that remains below g (g which finishes to be constant and equal to 3) except in rare but regular peaks (every 10^6) which go over g on a small interval (less than 10^{-6}).

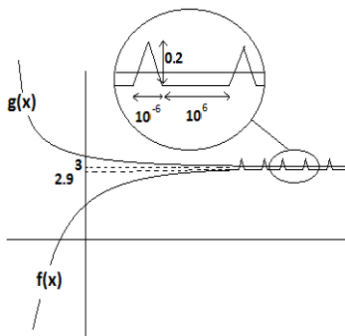


Figure 1 : The "monster"

Next, the teacher asks if "the monster" is an example, a counter-example or a 'hors-sujet' for the conjecture C3 ('hors-sujet' means that it does not fit the hypothesis). There is a debate and finally a student asks: "But what is the limit of the function f ?". This inquiry is not exactly a request for a definition, but the dynamic conception of the limit is starting to look fragile, so it seems that it would be time to go on to the second step: give a meaning to the formalization of the limit, formalization which will be given by the teacher in the end.

This second step is based on the notion of equality in calculus. Indeed, if in algebra the equality $A=B$ is usually translated by " $A=B$ if $A-B=0$ ", and in the set theory in terms of double inclusion " $A=B$ if $A\subseteq B$ and $B\subseteq A$ ", in calculus the equality between two numbers is usually translated by " $A=B$ if for every $\varepsilon>0$ the distance between A and B is less than ε ". And this equality can be applied to a function like: f verifies $P1$ if "there exists A such that for all x more than A and for all $\varepsilon>0$, $L-\varepsilon<f(x)<L+\varepsilon$ ". In these conditions, if f verifies $P1$ then exists A such that for every x more than A , $f(x)=L$. What is quite interesting is that if we exchange the quantifiers with $P2$ ($P2$: "for every $\varepsilon>0$ there exists A such that for every x more than A , $L-\varepsilon<f(x)<L+\varepsilon$ ") then we get the definition of the limit: f has got the limit L in plus infinity if f verifies $P2$. The didactic engineering is based on the idea that the study of $P1$ may be used to give a meaning to the variables A and ε as proximities of infinity and of the limit, and then, inverting $P1$ into $P2$ the formalization of the limit can be much better understood. Here we will deal with the study of $P1$ by the students. This study of $P1$ is divided into three phases. The first phase is a call for a conjecture. The property $P1$ is given by the teacher under the simplified form (to reduce the number of variables): $P1$: "for every real $x>50$, for all $\varepsilon>0$, $2-\varepsilon<f(x)<2+\varepsilon$ " and students are asked to say what can be concluded if f verifies $P1$. Three suggestions are made by them:

C3.1 : "If f verifies $P1$ then f is upper bounded"

C3.2 : "If f has got a limit and f verifies $P1$ then f is between $2-\varepsilon$ and $2+\varepsilon$ "

C3.3 : "If f verifies $P1$ then f is bounded by $2-\varepsilon$ and $2+\varepsilon$ since 50 "

The conjecture C3.1 is invalidated and the conjecture C3.3 is validated through the debates, whereas the study of the conjecture C3.2 is postponed.

For the second phase, the teacher asks to give examples of functions that verify $P1$. The first suggestion ($f(x)=2$) is soon validated. The second suggestion ($f(x)=2+\varepsilon$) is rejected by the teacher who will point out that he wants functions of a single variable x . Then a third and alternative suggestion is given ($f(x)=2+1/x$). This suggestion will be invalidated by a student taking $x=51$ and $\varepsilon=0.001$.

Finally, a fourth conjecture is made under the form of a drawn curve of a function f (see below).

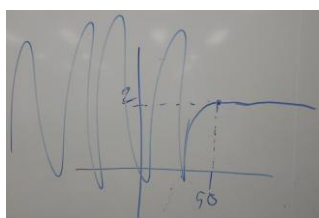


Figure 2 : Suggestion C3.4

The students' debate will soon validate this function f as an example of C3. For the third and last phase of the study of $P1$, the teacher gives the conjecture C4 and asks the students to study it: C4: "If f verifies $P1$ then for every $x>50$, $f(x)=2$ ".

In this article we are going to study the debate about the conjecture C4 in terms of rationality. And we will first give more details about this model of rationalities.

SOME TOOLS TO ANALYZE OUR CORPUS

In Boero(2006) the Habermas' rationality construct is used to analyze the production of a student. This rationality is split into three interrelated components: *epistemic rationality*; *teleological rationality*; *communicative rationality*. The kind of rationality we are using here is an epistemic one: we call rationality the means used by humans to try to understand a domain of reality, i.e. to try to avoid contradiction between what they may think and the reality they are attempting to figure out. Rationality achieves this aim of no contradiction by maintaining the coherence between three aspects: the kind of domain of study (reality or theory), the kind of logic (everyday logic or mathematical logic), and the kind of validation (abduction, induction, deduction, argumentation...: see Balacheff 1988, Pedemonte 2003, Meyer 2008. We will also split this epistemic rationality into three components. The first component, which we will call *pragmatic rationality*, using a kind of reality that does not need to be defined: it is here and it is so. Natural logic is usually used (implication is viewed as causality for example) and validation is linked to efficiency (it works or doesn't) or to obviousness. The second component, which we will call *empirical rationality*, seeks to produce general laws with systematic verification of cases. This empirical rationality can be of two kinds. The first one, inductivism, considers that a statement is true if it is true for many various cases. We can often see students explain that it 'must' be true because the studied property always seems to work or there seems not to be any counter-examples. The kind of truth associated with this rationality is plausibility. The second one, falsificationism, considers that a statement cannot be taken as true with only verified cases. The only certainty given by special cases is that the statement is false when they are counter-examples. Students using such rationality will declare that there is no objection for a statement for which no counter-example has been found. But they will also add that they cannot say that such a statement is true since it has not been proven. Usually, they use that kind of rationality because they are not able to prove it, and they are aware of that, or because they are only trying to find counter examples. The kind of truth is also plausibility. The third component, called theoretical rationality, uses the means of demonstration to prove statements. A statement is true if it can be proven by a demonstration. A student using this rationality tries to provide a proof using other properties, theorems, and definitions. These three components will help to see the different forward and backward movements of students' reasoning and attempts to understand, and will help to highlight the steps of the debate.

STUDY OF THE SCRIPT

The teacher gives little time to the students to consider if they think that the conjecture C4: "if f verifies P1 then for every real $x > 50$ we get $f(x) = 2$ " is true or false (P1: "for every real $x > 50$, for all $\varepsilon > 0$, $2 - \varepsilon < f(x) < 2 + \varepsilon$ "). Then a vote is taken. Twelve students think C4 is true, five think it is false, and nine vote "something else" (which

means that they have good reasons not to vote True or False. For example they do not understand the conjecture, or they think it is true and false, or they think it is true but they have no proof, or...).

Valentin starts to explain that he thinks it is false because he has a good candidate for a counter-example but he wonders if he has the right to take a function with two variables. The teacher says that the rule is that such a function is forbidden.

Then Erika, who thinks that the conjecture is false, comes to the board and draws what she considers as a counter-example.

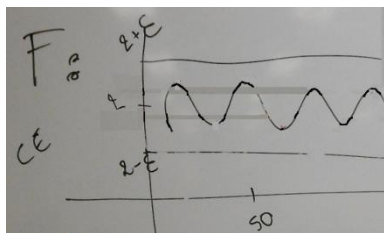


Figure 3 : The false counter example of Erika

Erika's suggestion can be taken as the statement: "the function drawn here verifies P1 because the drawing shows that the function is really between $2-\varepsilon$ and $2+\varepsilon$ ". We can say that this argument comes from a pragmatic rationality as it verifies only one value of ε . Five other students declare that they also think the conjecture is false for the same reason. Baptiste, who voted false, adds that he had another counter-example $f(x)=2 + \varepsilon \sin(x)$ but he gave it up because of the invalidation of Valentin's argument. He also adds that he is not sure about Erika's suggestion, concerning the "little epsilon". This new argument, the "little epsilon" is probably coming from an empirical rationality (falsificationism) which leads Baptist to imagine some other verification. Of course "little epsilon" is playing a special role in the problem and this new information will be helpful. Then Mathieu comes to the board to show that Erika's drawing does not verify P1 as "In P1 it's written for every epsilon, so we have the right to change the epsilon" (see fig. 4). This also corresponds to an empirical rationality, falsificationism, (maybe theoretical rationality but no demonstration of C4 is given) in the sense that the universal quantifier is highlighted: a general law is involved. Mathieu adds to the drawing:

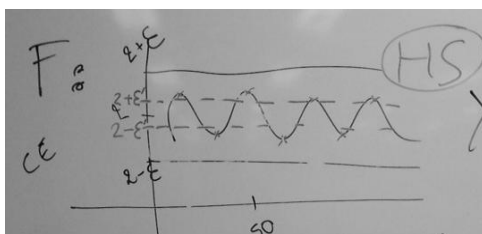


Figure 4 : Invalidation of the counter-example by Mathieu

Then, Louis, who thinks the conjecture is false, expresses his disagreement: we don't have the right to use the same symbol twice on the same drawing (Mathieu, like Erika, had used the symbol ε). So Thomas suggests using a prime (ε') but this option does not please Louis. Indeed, Louis explains that when one takes an epsilon which is

smaller, the function f still remains between the bounds because it decreases at the same time. In fact, Louis is mistaken because of an inversion of quantifiers. This can be linked to a lack of knowledge about the interpretation of a double quantification which is known to be difficult (Dubinsky & Yiparaki 2000) but also to an insufficient definition of the function by Louis, consequently to a pragmatic rationality. At this moment, the teacher explains that if the function decreases with epsilon it means that it is dependent on epsilon just as Valentin said.

Then Maxim wants to remind (wrongly) the class that number 3 and number 2,99999....(infinity of nines) are two different numbers (probably to build a new counter-example, but he does not say it). The teacher says that a proof had been done, supposing that they were different but in fact it had never been concluded that they were so. Maxime seems to get confused between the validity of an inductive reasoning and the truth of the hypothesis. This corresponds to a pragmatic rationality in which no absurd supposition is usually done to prove something. We will see that this question will come up again later with another reductio ad absurdum.

Quentin who thinks C3 is true goes to the board to make a drawing:

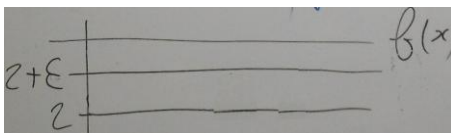


Figure 5 : The Quentin's attempt with a reductio ad absurdum

Quentin: To show that it's true I'm going to show that $f(x)$ can't be different from two. If it's not different from two, it's equal to two, do you agree? So if $f(x)$ is different from two, there are for sure some values between $f(x)$ and two. So P1 won't be verified. So the example, $f(x)$ different from two, is false.

Quentin gives a reasoning which is a reductio ad absurdum while taking a function which never equals two. This is an argument within a theoretical rationality. The desire to prove is clear. We will call this argument a "generic example" as Balacheff defined (Balacheff 1988). This kind of proof is used by students to prove a statement by taking an example, not for itself but as the representative of a class of elements. Here, the example is a constant function: but what is important is that the function is not equal to two. For students, this reductio ad absurdum is not easy to accept. There is the same confusion as with Maxime between the validity of the reasoning and the hypothesis.

Aya: Can you explain another time because I did not quite understand?

Elsa: Quentin, you voted True?

Quentin: Yes, I voted True.

Elsa: There's something wrong!

Nathan: In fact, you're saying it's False. You voted True, but you're saying it's false.

Elsa: Yes, I've got the feeling that it's an argument for False.

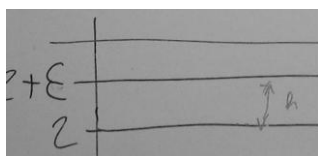
Margot: Yes, he's saying that False is false.

Teacher: We're going to listen to him once again because he's pretending that it's an argument of Truth. Go on Quentin, and listen to him.

Quentin: I'm saying that it's true for $f(x)=2$ by showing that $f(x)$ different from 2 is false. I'm showing that $f(x)$ cannot be different from two. (big silence)

Louis then asks "to develop Quentin's line of reasoning":

Louis: If we take the first function, we had $f(x)=2$. That's what we're trying to say. In fact, I will start like Quentin: if the function is different from two, it means that somewhere it makes a "jump"; this way it has at least one value different from two. So, for example, here! We can make it little or we can make it big. Here it is less than two minus epsilon. And if we take a value, here h , we will always find a value for epsilon which will be less than h , h which is the highest peak of the function which would be close to two.



This argument is clearly coming from a theoretical rationality. We will call it "mental experience" as Balacheff called this kind of proof (Balacheff 1988). We can see that in this proof there is some action ("jump", "make it big"), time is playing a role ("always"), and the proof is personalized ("we will"). All these are elements that characterize a proof called "mental experience". It is considered for Balacheff as a proof but still not a demonstration because it has to be depersonalized, decontextualized, and time has to be removed.

However, this proof gains unanimous acceptance: many hands go up to say that they want to change their vote. A last vote is organized: almost everybody votes True. A few remain who are not completely convinced.

The teacher will, then, call for a conjecture: What can we say of a function that verifies P2? (P2 which is the epsilon-delta definition of "f having a limit of 2"). And a new debate will begin...

TWO KEYS: RATIONALITY AND KNOWLEDGE

If we try to understand what underlies the students' queries and what leads them to the solution, we can see that it is the suggestion of arguments and counter arguments (use of rationalities) which slowly becomes deep and robust (knowledge) until a proof is valid and collectively accepted. We can note that the intervention of the teacher - apparently rather light - consists in fact in eliminating a tough problem: the inversion of quantifiers (there exists f such for every $\varepsilon > 0$... or for every $\varepsilon > 0$ there exists f ...) which will have to be faced (in fact it is supposed to be done after the study of P1 with the study of P2). So his intervention is crucial to allow students to

focus on the "right" problem. We will then especially note the decisive role that the counter-example of Erika plays, as it leads to make the *reductio ad absurdum* "concrete". Indeed, those who believe the conjecture to be true are thus forced to show that this counter-example in fact is invalid. So they are led to suggest a reasoning that invalidates this argument: they have to choose a "good" epsilon. This choice is in fact the second part of the *reductio ad absurdum*. Then Quentin suggests the negation of the hypothesis (which is the first part) but, as the conclusion includes a universal quantifier, he gives the negation with a universal quantifier also ($f(x)$ never equals two). Then Louis can "repair" that with a (more) existential quantifier. We can notice that Louis uses both kinds of language in his proof (semantic and syntactic): the "jump", "the "highest peak" but also "there is at least one value different from two" (for communicative rationality of Habermas, see Boero 2006). It is this second demonstration that leads to a collective agreement.

In the end, students can admit the irrefutability of the reasoning when all their reluctances have been taken into account by their peers. What seems decisive, in terms of rationalities, is that students can choose on their own the one they consider best adapted to the situation and to their own understanding of it, which could not happen if the teacher intervened. It gives them the opportunity to enter the problem with a pragmatic point of view, taking examples, to use falsificationism to test beliefs or to use a theoretical point of view for proving. It gives them also the opportunity to experiment the robustness of their own beliefs and potentially to modify them. Here, the shared culture of debating and, thus, of moving from a rationality to another, allows students to convince their peers or to be convinced by them. The collective reasoning of students becomes a progressive, rational and guided construction of syntactic and semantic elements that will constitute the bricks of the meaningful proof which is finally realized.

If rationality can lead to a good idea of limit, the reverse can be also observed: it seems clear that the concept of limit, with its unique complexity and proximity, can constitute a real opportunity for rationalities to be developed.

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