A model to analyse the complexity of calculus knowledge at the beginning of University course

Two examples: parametric curves and differential equations

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Our research focuses on the difficulties students encounter with the learning of calculus, considering that they have to cope with much more mathematical objects but also with new ways of reasoning – not only algebraic calculation, but also the practice of approximation, and a scaffolding way of using functions, limits, derivative, integrals, etc. for proving. The semiotic facet of new objects, and the way to manage it, is also a source of great difficulties. We use a model (Bloch & Gibel, 2011) to describe students' work when they have to deal with the resolution of exercises about parametric curves and differential equations.

Keywords: Calculus, students' understanding of mathematical signs and objects, reasoning processes, parametric curves, differential equations.

INTRODUCTION

Every researcher knows that mathematical work in the field of Calculus is usually very difficult for even good students when they are entering the University. We have studied the transition between the secondary mathematical organisation in teaching (pre)calculus, and the University one; in this perspective we aim at classifying the different 'things' students have to cope with when they are practicing Calculus.

The organisation at Secondary school takes into account some mathematical objects, as functions, derivatives, integrals: but a number of researchers underline the fact that the way these objects are introduced leads to algebraic calculation and not to analytic work. For instance, students are supposed to calculate an integral but not to justify why it exists; to study the variations of functions with derivative, but not to have a knowledge about which functions get derivatives at which points, or not. So we can see that the *raison d'être* of a mathematical concept is not highlighted.

We notice that, even if teachers think of the structural level, in most cases they confront students only with the operational one. For instance, Ghedamsi (2015) analyses a first year regular course at University and she concludes that:

> the teacher does not intervene to enrich (the work) by emphasizing relationships among notions, by changing the setting of semiotic representations, by leaving openings to organize knowledge, by making assessments of knowledge, etc. (So) students employ methods used at the secondary school and do not succeed to shift to the university ones.

In our case, the problem seems not to be the way the notions have been taught; we got an access to the students' course notes, and they show relevant justifications and explanations. The didactical repertoire of the class has been elaborated by suitable exercises and situations, leading to highlight the operating mode of these concepts. But to better explain the students' work, it is necessary to classify the objects, signs and reasoning processes they have to cope with during resolution of calculus problems.

I. THE LEARNING OF CALCULUS: OBJECTS, SIGNS AND REASONING

Mathematical objects and signs: complexification

At the beginning of University studies, students meet functions as in Secondary school like rational ones as polynomials, or sinus or cosine; they have to solve problems with exponentials, logarithms, but the derivatives can also generate new functions, and integrals too, or series: so objects may have different status, and signs become polysemic. With respect to these signs, we notice that at Secondary school students operate frequently by implementing isolated techniques: they can calculate on a rather straightforward way. But at University, they face complex signs and they have to associate different kinds of symbols, sometimes through a long proving process, for example to calculate an rather complex integral or to prove that a theorem is valid, which is not their responsibility at Secondary school. At University too, signs are multiform: for instance derivative can be written *f'* but also *df/dx*; or *x* can be the function, so it will appear as *dx/dt*; a letter can nominate a variable, a function, or a parameter, which status is sometimes difficult to understand.

Moreover, rules about the use of signs are imbricated, so if you try to calculate $\int cos^2x\, dx$ you have to linearize cos^2x because you cannot apply the rule of the primitive of x^2, just 'mixed' with the primitive of *cos x*, to cos^2x… and find $1/3.sin^3x$, as we saw once a student. This evolution of signs is even more evident considering the procedures for proving within the calculus work: students have to understand and use new analytic methods, as it is well known, for limits with ε and α, and to master quantifiers, which reveals to be rather hard (see Chellougui & Kouki, 2013).

Reasoning processes

This complexity requires that students adapt themselves to improve and perfect their reasoning processes: they have to become able to deal with all the facets of knowledge and to adapt their "way of doing", taking into account all the aspects of a question and the requirements of the proofs.

We can say that throughout the reasoning processes, signs (and then objects) work in a strong interaction, as seen above: integrals with the primitive of sine and squares, but also techniques and technologies to prove. Among these technologies it is very important that students learn how to manage the new tools, as quantifiers and the way to perform a valid reasoning up to its end.

II. A MODEL TO ANALYZE STUDENTS' PRODUCTIONS

We need then a tool for modelling students' reasoning processes and try to seize how they manage with this complexity at each level of a situation. This tool takes its origin in the TDS (Theory of Didactical Situations, see Gonzalez-Martin, Bloch, Durand-Guerrier, and Maschietto 2014). Let us recall that TDS is trying to implement situations with an adidactical component, that is, situations that allow students to live a heuristic phase of research. Then they can validate their conjectures through a confrontation to the elements of an adequate milieu. Eventually, a phase of institutionalization is managed by the teacher. The whole model can be found in Bloch & Gibel, 2011.

The theoretical tools used in the elaboration of the model

We want to take into account the semantic dimension – the *meaning* of the aimed knowledge – to analyse reasoning processes: this contributes to justify our choice of the TDS as a basis of our model. TDS organizes adidactical situations with three phases (corresponding to levels of the milieu): a heuristic one (students' action) grounded with a question; a formulation and validation one; and a last one, institutionalization by the teacher. In this configuration reasoning processes we take into account are as well valid or erroneous ones. This theoretical frame allows developing also an analysis of the *functions* of the reasoning processes within the situation (Gibel, 2004; 2015). So in our model we consider signs, functions of reasoning, and levels of argumentation.

The semiotic dimension of the analysis

In order to complete and enhance this theoretical framework we add a semiotic content to TDS. In a previous research (Bloch & Gibel 2011; Gibel 2015) we highlighted the fact that reasoning processes elaborated by the pupils and the teacher during a lesson can occur in various ways: linguistic, calculative, scriptural, and graphic elements (see also Bloch 2003). Consequently the semiotic analysis constitutes one of the dimensions of our model, completing those previously presented: on the one hand the function of the reasoning processes and on the other hand the corresponding level of the didactical milieu. Let us notice that signs can be either formal or linguistic: both will be taken into account. What is significant are the arguments embodied in those signs. This is why Pierce's semiotics seems particularly appropriate for our research and will enable us to study more precisely the evolution and the transformations in the signs used by the different actors within the situation.

In our application of Pierce's semiotics we use the three usual designations: icon, index-sign and symbol-argument. Yet we do not consider the whole intricacy of Peirce's theory: it would be too complex to take into account and not necessary to correctly interpret students' actions in the situation. So we just correlate icons with students' intuitions, drawings, examples, resolution attempts; indexical signs with local proofs, first tools for validation, more accurate reasoning, formulations of mathematical objects; and symbols-arguments with the concluding validation and mathematical formulation of the rules, and of the aimed knowledge.

The didactical repertoire and the repertoire of representations

The work in the students group leans first on the existing *repertoire*: all the semiotic means used by a teacher, and those he expects from his pupils through his teaching, establish the didactical repertoire of the class – as defined by Gibel (2004). The didactical repertoire of the class can be identified as being part of the mathematical knowledge the teacher has chosen to explain, namely during validation and institutionalization phases of previous situations or previous lessons. The repertoire of representations is a constituent part of the didactical repertoire. It is made up of signs, diagrams, symbols and shapes and also linguistic elements (oral and/or written sentences), which make it possible to name the objects encountered and to formulate properties and results.

A model to analyse reasoning processes

The model of structuration of the didactical milieu used in this construction is that of Bloch (2006). The chart below (Table 1) sums up the levels of milieu – from M1 to M-3 – corresponding to the *experimental* situation.

M1 Didactical milieu	E1: reflexive subject	P1: P. planner	S1: sit. of project
M0 Learning milieu : institutionalization	E0: generic student	P0: professor teaching	S0: Didactical situation
M-1 Reference milieu : Formulation and validation	E-1: The subject as learner	P-1: Professor Regulator	S-1:Learning situation
M-2 Heuristic milieu : action, research	E-2: The subject as an actor	P-2: P devolves and observes	S-2: Situation of reference
M-3 Material milieu	E-3: epistemological subject		S-3: Objective situation

Table 1 *–Structuration of the didactical milieu*

The negative levels are of particular interest in the sequences we frequently study since they allow describing the emergence of a proof process in the setting up of an adidactical situation. The place where we hope to see the expected reasoning processes appear and develop is located at the articulation between the heuristic milieu and the reference milieu.

In our previous research (Bloch & Gibel, ibid.), we decided to focus our didactical analysis on three main axes to study the reasoning processes. The first axis is linked to the nature of the situation: in a situation involving a research dimension, students produce reasoning processes which depend to a great extent on the involved phase of the situation, that is, the level of milieu (heuristic milieu, milieu of formulation or validation) (Table 1).

The second axis of our study is the analysis of the functions of reasoning. We aim at linking these two axes, showing how the reasoning functions are linked specifically to the levels of milieu and how these functions also *manifest* these levels of milieu.

The third axis concerns noticeable signs and representations. These elements can be observed through different forms which affect the way the situation unfolds.

The application of this model to a situation will then include an analysis of the milieu and semiotic analysis of the students and teacher's productions. We will interpret the conjectures, intuitions, signs and reasoning processes as an evolution of the didactical repertoire of the class, knowing that the situation aims at developing a mathematical knowledge in the field of calculus. This is summarized in Table 2:

	Milieu M-2 Heuristic level	**Milieu M-1** Formulation, validation	**Milieu M0** Institutionalization
Nature and functions of reasoning	R1.1 SEM - Intuitions on a drawing - Decision of calculation - Heuristic tools; errors - Exhibition of an example /a counter ex.	R1.2 SYNT/SEM - Generic calculations and conjectures (right or wrong) - Decision on a mathematical objet	R1.3 SYNT - Formalization of proofs within the mathematics involved theory
Level of use of symbols	R2.1 SEM Icons or indices depending on the context (schemas, intuitions…)	R2.2 SYNT/SEM Local or more generic arguments: indices, calculations	R2.3 SYNT Formal and specific arguments: symbols *hypoicons*
Actualisation of the repertoire	R3.1 SYNT/SEM - Ancient knowledge - Enrichment at the heuristic level: calculations, conjectures	R3.2 SYNT/SEM Enrichment at the argumental level: - statements, reasoning	R3.3 SYNT - Formalized proofs - Signs within the relevant theory - theoretical elements

***Table 2** – A model to analyse situations*

Table 2 then includes levels of milieu, nature of signs, functions of reasoning, level of the repertoire. We have also pointed out that some formulations are made on a semantic mode (SEM), as more evolved (in a mathematical sense) ones can also be formulated on a syntactic mode (SYNT). Let us notice that this model allows not only the study of adidactical situations, but also to analyse students' productions while solving 'ordinary' problems: in this text we choose to develop this feature of our work (see examples in III.). The matrix notation R1.1 etc. allows to quickly situating the level of arguments where students are located.

We want to underline the fact that an *a priori* analysis is necessary for each situation we choose to study: the model we built is also useful and efficient to perform this *a priori* analysis, as it allows anticipating resolution processes and difficulties. In this perspective, we classify reasoning, calculations, formulas, the nature of signs produced, and knowledge(s) expressed by students in the different phases, reflecting the situation in which they are located.

The use of the model to analyse 'ordinary' teaching

Our model can also be used to analyse 'ordinary' secondary or university teaching, as it allows detecting students' reasoning processes, use of symbols, and understanding of mathematical objects involved. We can analyse the students' templates while they try to solve a problem: they are first in a heuristic milieu M-2, trying to find a resolution process. Then they decide to undertake calculations, use of theorems, and a final issue. They are then in a milieu M-1. This can be seen also in the context of an evaluation which we present in part III.

III. TWO EXAMPLES: HOW STUDENTS COPE WITH PARAMETRIC CURVES AND DIFFERENTIAL EQUATIONS

We consider the productions of fourteen students in the context of a first year terminal exam at the University of Pau, in May 2014. The teaching unit involved is named: "Mathematics of the movement", which is interesting because a link is made between mathematical knowledge and physics problems. The exam includes three exercises, the first one on polar and parametric functions, the second and the third ones on differential equations. Parametric curves and differential equations are especially interesting to study as they involve complex new signs, unusual processes for secondary students, and new kinds of reasoning. These reasoning encompass also mathematical objects, as functions, limits, derivatives, but in a new way of thinking.

1. Parametric curves

A parametric curve is of the type: $x = f(t)$, $y = g(t)$. There are two functions x and y to study; students must understand that what is required finally is to describe the variations of y with respect to x, in the case of a movement for instance; so the study of the two functions f and g (including the calculation of their derivatives) is just a step (of R1.2 type) to interpret what happens with the curve of y while x being the final variable. Sketching the graph needs to give values to t, being sure that we got the 'whole' curve; or eliminating the parameter t, which may reveal to be complex.

Another difficulty comes from the existence of tangents: in contrast to what happens with algebraic curves, parametric ones can have two tangents at the same point: this is a singular point that students did not meet before. They are expected to identify the nature of this singular point, for instance a cusp. It needs to first apply a formula ($x'(t)=0, y'(t)=0$) and then try to find the tangents at this point to be able to identify the nature of the singular point (a calculation and reasoning of successive derivatives that takes place at R2.3 or R3.2 level at least and involves specific interpretation about the objects at stake).

We classified students' productions from S1 to S14. In May 2014 students were confronted to the following question:

Let us study the parametric curve defined by $x(t) = a\, t^2/(1+t^2)$ *,* $y = a\, t^3/(1+t^2)$ *with* $t \in \mathbb{R}$*. Show that it is sufficient to study for* $t \geq 0$*. Determine the variations and confirm that the curve gets symmetry, an asymptote and a singularity.*

Students have to calculate $x(-t)$ and $y(-t)$ and conclude about the kind of symmetry; calculate the derivatives, build the variation table and do not forget the limits; and they must undertake pertinent interpretations of these results. The curve has a singularity, a cusp: they must find its coordinates and its nature. We expect that a difficulty can occur in the interpretation of derivatives: students are accustomed to calculate such derivatives but for algebraic functions one derivative is enough to find the variation of *f*. The asymptote can be a problem too, as $t \to +\infty$ when $x \to a$ and $y \to +\infty$. So the asymptote is vertical, but nevertheless when $t \to +\infty$, which can be a source of misunderstanding: for algebraic curves a limit where the variable tends to infinite corresponds to a horizontal asymptote.

Analysis of students' productions

Student S1 does perfectly all what is expected: she calculates the derivatives, the behaviour of the function, draws the graph with the asymptote, and determines the cusp with its tangent, which needed to calculate $x^{(3)}(t)$ and $y^{(3)}(t)$ for $t=0$. S1 reaches the level R1.3, she makes a formalization of proofs within the required theory. Student S14 cannot do anything; five other students encounter difficulties to calculate derivatives, to interpret the symmetry, and to find the singular point. One student says that *a* should be the parameter. One other writes that the equation of the curve is $x(t)+y(t)$... So we can see that even in M-2, some students do not reveal to be able to undertake local adequate calculations, as they do not understand that they are no more in the case of a Cartesian function. There are errors about the nature of the asymptote, for instance: only six students calculate the limits and conclude about the asymptote, reaching the R2.2 level, but among these six two of them write a wrong equation: $y=a$ instead of $x=a$. Students' productions also show calculation mistakes, especially in derivatives and primitives. The handling of singular points is not properly integrated: students are *unsettled* with the conditions for being a singularity, with the ways of finding the tangent... For instance S6 tries to find the point by calculating $x=0$ and $y=0$ instead of their derivatives; S2, who succeeds in the exam, writes that: "every non collinear vector to the curve is tangent to the curve"…

Some students who calculate without mistakes encounter problems with the interpretation of their calculations: their use and interpretation of signs do not exceed the R2.1 or R2.2 level. Those who succeed very well (four from the twelve) write sentences to explain that a singular point is given by $x'(t)=0$, $y'(t)=0$, applying a R2.2 or even R2.3 knowledge; one student says that it means that the speed is equal to zero; but only the first one S1 is able to calculate the tangent and identify the nature of the singularity, being clearly in the position R2.3 for all needed symbols.

2. Differential equations

In the exam students had to cope with the solving of these two differential equations:
Exercise 3: Given the first order differential equation: $e^x yy' - x^2(y^2-9) = 0$
After separating the variable, solve the equation. Then solve the Bernoulli differential equation:

$$y' - \frac{4}{x}y - x\sqrt{y} = 0$$

A priori analysis

First, we consider the first order differential equation. Separating the variables implies preserving the initial shape, that is, not to develop the term $x^2(y^2-9)$, to obtain the following shape: $\frac{yy'}{y^2-9} = \frac{x^2}{e^x}$. This requires analysing preliminarily the features, the characteristics of the different mathematical signs appearing in the equation to anticipate the expected form. To solve this equation, students have then to transform y' as $y' = \frac{dy}{dx}$; then they can produce an algebraic form allowing to integrate the terms.

Dealing with the term $\int \frac{x^2}{e^x}dx$ requires necessarily applying *two times* integration by parts. Considering the second part of the exercise, solve the Bernoulli equation gives rise to a number of difficulties: the first one consists in being able to make the substitution leading to the equation $2zz' - \frac{4}{x}z^2 = xz$. After simplification it can then be written: $2z' - \frac{4}{x}z = x$.

Students must solve first the homogeneous differential equation associated, and then they have to solve the inhomogeneous differential equation by variation of the constant, which can be source of new difficulties. The technique of variation of the constant is a part of the new technical and technological tools of first year University course, so it is of Level R3.3 in our model.

Analysis of students productions

First we analyze main difficulties encountered by students to solve the differential equation $e^x yy' - x^2(y^2-9) = 0$. The first one is to separate the variable to obtain $\frac{yy'}{y^2-9} = \frac{x^2}{e^x}$: among fourteen students only eight of them accomplished this task; for two of them this task was difficult and required several attempts as we expected. The next step of the resolution needs to compute $\int \frac{y}{y^2-9}dy$. Seven students out of eight were able to fulfil this task, but two of them represented the quotient as a sum of rational functions, because they did not acknowledge the derivative of the function

$Ln(y^2 - 9)$. Recognize this primitive is of Level R2.2 because students have to identify a schema – a hypoicon according to Peirce – of different 'models' of derivatives/primitives, which variable is not always '*x*'. It supposes that the students' repertoire encompasses a lot of 'forms' that at this level they did not meet often enough.

We notice that only five students were able to deal with the term $\int \frac{x^2}{e^x} dx$ applying two times integration by parts. Then, only four students resolved this equation and obtain the whole solution.

As regards the Bernoulli equation, half of the students recognized an equation such as $y' + a(x)y = b(x)y^n$, with $n = \frac{1}{2}$ and $a(x) = -\frac{4}{x}$, $b(x) = x$. They have been able to make the substitution $z = y^{\frac{1}{2}}$. But only five of them succeeded in obtaining $2z' - \frac{4}{x} z = x$; two students did not allow themselves to reduce the equation, they could not admit the possibility of dividing each term by z. Among these five students, the three other students implemented successful method of solving.

CONCLUSION

We can conclude that it is really difficult for students to access to Level 3 of our model, although this level being the 'expert' one required: they frequently keep blocked at Level 1 with old non-adapted knowledge or false calculations, or they try to work at Level 2 but do not succeed in more complex calculations, especially when schemas are involved; or they make the expected calculation but are no more able to interpret it within the problem.

We can notice that the involved activities, at this level, imply a very rich assortment of technics, procedures, and a variety of occasions to apply formulas. Yet the familiarity with this knew field of knowledge is not established for a majority of students, and their algebraic skills are undersupplied. Then the students' productions highlight their numerous attempts to try to calculate and recognize well-known shapes within the heuristic milieu. We think that the difficulties highlighted in this study are not linked with the teacher's didactical choices, but they are common within the population of mathematical students, due to the reasons we evoked in the first part of this paper.

We want to point out the missing knowledge also in the (French) secondary curriculum: students study no more the composition of functions. Yet recognize the kind of schemas we see in a differential equation as above implies to detect which functions are at stake and how they appear in the formula. As students have no familiarity with 'the whole formula' they try to interpret each element separately, which has no meaning. So, most of them do not achieve the level R3.

We could also formulate these obstacles by saying that students fail in doing a pertinent association between syntactic and semantic methods: they are stressed with calculations and cannot control the meaning of the operations they have done. The next step of our work should be finding relevant situations for the teaching of Calculus, both in an introductory way at Secondary school, and at University.

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