Procedural and Conceptual Understanding in Undergraduate Linear Algebra

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This paper discusses the learning of concepts in undergraduate Linear algebra by pre-service teachers in mathematics. The focus is set on the bi-linear and multi-linear forms on a real vector space, exemplified by the dot product of vectors and determinants, respectively. Moreover, the paper identifies and describes discrepancies between students' achievements regarding the development of procedural and conceptual understanding. They are investigated through two types of exercises, discussing questions and multiple-solution tasks (MSTs), whose solutions differ under three criteria.

Keywords: linear algebra, pre-service teachers, conceptual understanding, multiple-solution tasks.

INTRODUCTION

This paper elaborates the current state of research about procedural and conceptual understanding. It focuses on a content-specific domain about linearity, bi-linearity and multi-linearity in undergraduate Linear algebra. In particular, I investigate pre-service teachers' understanding of these concepts. Further on, I argue that in order a task to be called a multiple-solution task (MST), a minimum of *three concrete criteria* for the diversity of the solutions must be fulfilled. These criteria may vary in different domains of mathematics. In this paper, I try to specify the definition of MSTs by Levav-Waynberg & Leikin (2009), by giving such criteria in the field of Linear algebra.

THEORETICAL FRAMEWORK

The theoretical framework consists of two parts, one referring to research on procedural and conceptual understanding, and, two, dealing with bi-linear and multi-linear forms in university and high school mathematics.

Procedural and Conceptual Understanding

Procedural knowledge, as defined by Hiebert & Lefevre (1986), is consisted of two parts: one, the symbolic mathematical language, and two, the "rules, algorithms or procedures used to solve mathematical tasks" (Hiebert, 2013, p. 3). The authors describe these procedures as subsequent step-by-step instructions that need to be executed when solving a mathematical task. One kind of such procedure is "a problem-solving strategy or action that operates on concrete objects" (Hiebert, 2013, p. 7). *Conceptual* understanding relates to a web of knowledge and is developed through an establishment of many relations between pieces of information or between existing and new knowledge. It does not have a linear sequential character. Hiebert & Carpenter (1992) explain conceptual understanding as a structured network of concepts, their representations, and properties [1]. In this paper, I would like to specify these two types of understanding according to the content domain of Linear algebra by giving three examples. Namely, knowing how to carry out the Gaussian algorithm can be seen as a procedural understanding and applying it to solve a system of linear equations or to find an inverse of a matrix, thus linking it to other concepts, can be considered as a conceptual understanding. Likewise, procedural knowledge of the dot product of two vectors is the ability to calculate it according to a formula involving the components of both vectors, while a conceptual understanding is a possibility to connect it with projections of vectors and the trigonometric function cosine, so to interpret the obtained scalar geometrically (Donevska-Todorova, 2015). Similarly, procedural understanding of determinants is knowing how to calculate them by the Laplace (cofactor) expansion, for example, and a conceptual understanding of determinants means knowing how to use them for determining the existence of an inverse of a matrix or to interpret them as oriented volumes (Donevska-Todorova, 2012).

Bi-linear and Multi-linear forms in University and School Mathematics

In this section, I refer to concept definitions of bi-linear and multi-linear forms in pure mathematics and I exemplify them by the concepts of the dot product of vectors and determinants, respectively. Afterwards, I discuss the treatment and the importance of the term *linearity* form a didactics point of view.

Definition 1: A *multi-linear form* on a vector space $V(F)$ over a field F is a function $f: V(F) \times \ldots \times V(F) \to F$ that satisfies the following axioms:

1. $\alpha \cdot f(u_1, \ldots, u_i, \ldots, u_n) = f(u_1, \ldots, \alpha \cdot u_i, \ldots, u_n)$
2. $f(u_1, \ldots, u_i, \ldots, u_n) + f(u_1, \ldots, u_i{}', \ldots, u_n) = f(u_1, \ldots, u_i + u_i{}', \ldots, u_n)$

for every $\alpha \in F$ and $u_1, \ldots, u_n \in V(F)$ and any index i. For $n = 2$, the form is called *bi-linear*.

For example, the function $f\big((x_1, x_2), (y_1, y_2)\big) = x_1 y_2 + x_2 y_1$ is a bi-linear form on $\mathbb{R}^2$ and the determinant of a square matrix of degree n is a n-linear form of its columns or rows.

Bi-linear and multi-linear forms can take values in any vector space since the axioms make sense as long as vector addition and scalar multiplication are defined. Yet, in bachelor studies for pre-service teachers we usually discuss the bi-linear, i.e. multi-linear form on a real vector space, as is also the case in this study. Bi-linear, i.e. multi-linear forms over other fields, e.g. of complex numbers, are not part of this study.

Theorization of *linearity*, e.g. classification of bi-linear and multi-linear forms contributed to the development of the unifying and general theory of Linear algebra (Dorier, 2000). The term *linearity* is one of the central terms in Linear algebra, for the reason that, it refers to linear combinations, linear (in)dependencies, linear mappings, bi-linear forms, such as scalar products and multi-linear forms, such as determinants, however, “linearity has not become an organizing idea for the students and this seems also to be true for quite a few teachers” (Tietze, 1994, p. 49). The term *linearity* is also used in high school, e.g. linear functions are studied in lower, and then, differentiation and integration in upper secondary education in relation to topics in Calculus. In high school Linear algebra, we teach linear transformations and treat only the concept of bi-linearity, though not the concept of *multi-linearity*. The term *bi-linearity* itself is never explicitly mentioned, nevertheless implicitly studied through the dot product of vectors. This is relevant for the transition from upper high school to university.

Exemplary research studies with a focus on students’ understanding of linear combinations (Possani, 2013) and linear (in)dependence (Bogomolny, 2007) have used different theoretical frameworks. However, it also seems that there is a lack of studies regarding the teaching and learning of bi-linear and multi-linear forms at any level of education.

DISCUSSING QUESTIONS AND MULTIPLE-SOLUTION TASKS

Before I proceed with elaborating the discussing questions and multiple solution tasks, I show a definition of determinants, which was applied during the observed lecture.

Definition 2. The mapping $det: \mathbb{R}^{n\times n} \to \mathbb{R}$ is called a determinant if the following hold:

D1: det is linear in every row.

D2: $rgA < n, det(A) = 0$

D3: $det(E_n) = 1$.

The definition axiom D1 means that both of the axioms 1. and 2. in the Definition 1 hold.

Discussion questions

Discussing questions in mathematics education offer a possibility for the students to talk and oral communication may contribute to the investigations concerning the development of procedural and conceptual understanding. When students articulate their thinking orally and by writing, they recall, reflect and consolidate their knowledge, and adequate understanding of concepts develops.

Discussing questions which were used in this study were the following.

Decide whether the following statements are true or false and provide argumentation to support your answer.

$(\forall A, B \in \mathbb{R}^{n\times n})$ True or false:

a) $AB \neq BA$, but $det(AB) = det(BA)$

b) $det(A + B) = det(A) + det(B)$

c) $det(A) = 0$ then A^{-1} does not exist

d) For $A \in \mathbb{R}^{2\times2}, det(A)$ is the oriented area of the parallelogram spanned by $A\overrightarrow{e_1}$ and $A\overrightarrow{e_2}$.

e) $det(A) = 1$if and only if $A = E_n$

f) $det(A) \neq 0$ if and only if for all $b \in \mathbb{R}^n$, the system of linear equations $A\vec{x} = \vec{b}$ has exactly one solution $\vec{x}$.

g) $det(AB) \neq 0$ if and only if A, B are regular

h) $det(A) = 0$ if and only if $KernA = \{\vec{0}\}$

i) For $A \in \mathbb{R}^{3\times3}, det(A)$ is the volume of the parallelepiped spanned by $A\overrightarrow{e_1}$, $A\overrightarrow{e_2}$ and $A\overrightarrow{e_3}$.

These discussing questions include many concepts in Linear algebra such as matrix, identity matrix, square matrix, invertible matrix, singularity/ non-singularity of a matrix, linear dependence/ independence of vectors, kernel, systems of linear equations, area and volume. Consequently, this wide net of concepts makes them suitable for explorations of both procedural and conceptual understanding.

Multiple Solution Tasks

The mathematical problems which were implemented in this study can be classified as *Multiple-Solution Tasks* (MSTs) (Leikin, Levav-Waynberg, Gurevich & Mednikov, 2006; Leikin & Levav-Waynberg, 2007) because there exist multiple paths towards their solution. Regarding MSTs in Linear algebra, these solutions may be diverse in the sense of different usage of

(1) *modes of description and thinking of concepts in Linear algebra* (Hillel, 2000; Sierpinska, 2000),

(2) *properties of concepts in Linear algebra*, and

(3) *subject-specific strategies or solving tools in Linear algebra*.

I now explain these *three criteria*. First, Hillel (2000) distinguished between *three modes of description* of concepts in Linear algebra: *geometric, algebraic* and *abstract*; and further on, Sierpinska (2000) described *three modes of thought: synthetic-geometric, arithmetic* and *analytic-structural*. Second, properties of concepts may be used for defining them, which is a usual way at university Linear algebra, or for describing them, which is typical for school mathematics. Third, in a concrete MST, it may happen that each subject-specific strategy for one of its solutions corresponds to exactly one mode of description and thinking. It may also be the case that different solutions require different solving strategies which all correspond to the same mode of description and thinking. Arguing through subject-specific strategies and exchanging geometrical and algebraic ideas and vice versa is a powerful tool for problem solving and obtaining deep understanding (Tietze, 1994). All three criteria are closely connected to both the procedural and the conceptual understanding. An example of such MST according to the three criteria, which is also a part of the discussion in this paper later, is given in Table 1. Yet, such criteria, which make a particular solution diverse from another, do not explicitly appear in the exemplary MSTs by Levav-Waynberg & Leikin (2009) or by Leikin (2007).

MST1: Find the determinant of the matrix $M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Write as many solutions as you can.	(1) Mode of Description/ Thinking	(2) Properties of Concepts	(3) Subject-specific Strategy for Problem Solving by the Use of:
Solution 1: $detM = det\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \stackrel{\text{def}}{=} 2 \cdot 2 \cdot 2 \cdot det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \stackrel{\text{def}}{=} 8 \cdot 1 = 8.$	Abstract/ Analytic-structural	Multi-linearity (Homogeneity axiom 1 in the Definition 2)	the axioms D1 and D3 in the Definition 3
Solution 2: $detM = det\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2 \cdot 2 \cdot 2 = 8$	Algebraic/ Arithmetic	The diagonal property for a triangular matrix	elementary matrix transformations
Solution 3: $detM = det\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$ $= 2 \cdot 2 \cdot 2 + 0 \cdot 0 \cdot 0 + 0 \cdot 0 \cdot 0 - 0 \cdot 2 \cdot 0 - 2 \cdot 0 \cdot 0 - 0 \cdot 0 \cdot 2 = 8$	Algebraic/ Arithmetic		Sarrus rule
Solution 4: $detM = det\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$ $= 2 \cdot det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - 0 \cdot det\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} + 0 \cdot det\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} =$ $= 2 \cdot 4 = 8$	Algebraic/ Arithmetic	A determinant of $n \times n$ matrix is as a sum of determinants of n sub-matrices $(n-1) \times (n-1)$	Laplace (cofactor) expansion
Solution 5: The determinant of M is equal to volume of 8 cubic units of a parallelepiped whose sides are obtained when each side of the unit cube is stretched twice.	Geometric/ Synthetic geometric		Geometry and linear transformations

Table 1: Example of a Multiple Solution Task in Linear Algebra

The aim of this study was, however, not to ask the students to explicitly provide more than one solution to all problems, but to ask them to offer one, which according to them, is the most rational in the number of undertaken steps and in the time required. I considered such written solutions, in addition to the oral responses on the discussing questions, as sufficient sources for analyzing students' procedural and conceptual understanding in the frame of this study.

RESEARCH QUESTION AND METHODOLOGY

Research Question

What kind of understanding do pre-service teachers show when they learn determinants during an undergraduate course of Linear algebra and analytic geometry?

The investigations aiming to offer answers to this question relate to procedural or conceptual understanding in the way they were described above.

Research Methodology

The research study took place at the Institute of Mathematics at the Humboldt University in Berlin. In their second semester, pre-service teachers take the course Linear algebra and Analytic geometry II. During this course, they study about the dot product of vectors and determinants, among other concepts. The goal of the undertaken observations was to locate and describe types of understanding which these students develop. The *observational protocol* (Creswell, 2013) includes researcher's notes about all undertaken observations and meetings with the lecturer, three teaching assistants and two tutors, who were responsible for the course. *Researcher's notes* consist of *demographic information* (time, place, date and participants), *descriptive notes* (instruction materials from the lecturer and the teaching assistants, and students' assignments) and *reflective notes* (reconstructions of dialogues, discussions and activities, researcher's personal thoughts, detections, ideas, proposals and impressions). The researcher neither took part in the selection of the exercises nor participated in the discussions during the lectures and the exercises sessions. In this way, researchers' influence on the teaching and learning process was eliminated. All information gathered by the observational protocol represents primary material to be analysed further on.

FINDINGS AND DISCUSSION

There are 120 students taking the course. After every lecture, they participate in course exercises and write home assignments each week. I first shortly discuss students' performance on the discussing questions which were part of the course exercises and then on the MST1 (Table 1) which was given as a homework problem.

During the actual learning process, students were allowed time to think about the true/false questions and discuss in pairs, before they articulate their thinking aloud. At least half of the students have stated their opinion about the validity of each of the statements (with an exception of one statement). In the space constraints of this article, I comment only two out of all nine discussing questions, which I consider were problematic for the students. Firstly, although the majority of the students claimed falsity of the statement b), none of them could give reasons why the statement is incorrect. Neither could they offer examples to show it. This illustrates students' uncertainty about the distinction between operations with matrices and determinants. In this statement, the multi-linearity property of determinants, so axiom D1 in Definition 2, more preciously axiom 2 in Definition 1 comes into focus and it seems that students were not able to explain it orally. Secondly, on the statement d) only three students gave answers, two of them claiming correctness and one of them falseness of the statement. This question is related to geometric visualization of determinants. What seems to be difficult for the students to comprehend is the establishment of a link between the algebraic symbolism of determinants and their geometric interpretation as oriented area of parallelograms or oriented volume of parallelepipeds (statement i). Moreover, it seems that students confused orientation, i.e a property of determinants changing their sign when any two rows (or columns) switch their places, with elementary matrix transformations (a conclusion derived from the observational protocol). This confusion is classified as a misconception in Linear algebra by undergraduate students (Aygor & Ozdag, 2012).

Out of the five alternative solutions on the MST1 (Table 1), surprisingly, students used only three (Solution 2, 3 and 4), none of them referring the definition axioms or geometry. This shows that, according to criterion (1), students used only arithmetic-algebraic modes of description in their written assignments, by applying the Saruss rule, the Laplace expansion and transformations of matrices, as subject-specific strategies for solving the problem, which meets criterion (3). According to criterion (2), it seems that students did not use the properties which construct the axiomatic-structural definition, rather others, e.g. determinant of a triangular matrix equals the product of its diagonal entries.

In addition to these findings based on the discussing questions and the MST1, I discuss one more MST [2] which it is the following.

MST2. Find the determinant of the matrix $D = \begin{pmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 2 \end{pmatrix}$.

There exist several adequate ways of solving the task. I consider the one based on the definition axiom D2 (in the Definition 2) the fastest because it is an argument on which I can immediately derive zero value of the given determinant. Analysing students written performance on this task (which was part of the observational protocol), I found out that only 44% of them used this definition. 28% of them used the Laplace expansion and the rest of the students used matrix transformations. These data, again according to the same criteria (1) and (3), show that students based their solutions on the algebraic mode of description (Hillel, 2000) the arithmetic mode of thinking Sierpinska, 2000), but not on the abstract-structural or geometric mode. Similar results were derived by analysing students written works on other problems [3].

The discrepancy between the utilization of the algebraic mode of description and the arithmetic mode of thinking on the one hand; and the geometric and the abstract-structural modes of description and thought, on the other hand, shows students' predominant possession of procedural versus conceptual understanding. In relation to the research question, it seems that, students think of the Laplace expansion as a secure way towards a correct solution, by carrying out a step-by-step sequence of calculations. In contrast to this, the geometric solution does not require computing skills, but visualizations and interpretations; and the abstract-axiomatic one, necessitates decision making and justification, which seem to be cognitively more difficult processes. This means that the students easily accomplish procedures, but face difficulties in shifting between different modes of descriptions (Donevska-Todorova, 2014), changing strategies and connecting more concepts (Donevska-Todorova, 2012a). Linking procedural and conceptual understanding can be accomplished by developing meaning for symbols and applying procedures to solve problems effectively (Hiebert & Lefevre, 1986).

CONCLUSIONS

The findings of this study show that pre-service teachers taking an undergraduate Linear algebra course have some problems when they learn the concept of *multi-linearity*. It seems that they do not understand what does linearity in a row (column) mean. This conclusion derives upon their insufficient argumentation and exemplification about the additive axiom 2. in Definition 1 discovered through the discussing questions; and absence of usage of the homogeneity axiom 1. in Definition 1 when solving MST1. Homogeneity and additive properties of determinants are often confused with matrix operations when students multiply by a scalar or add all entries of the determinant instead of entries in a single row (column).

The investigations on the performance on the MSTs, according to the three criteria, show that students prefer one mode (the algebraic mode of description and arithmetic mode of thinking), a few concepts' properties (not those which construct the axiomatic concept definition) and a few subject-specific strategies (calculi-based procedures, e.g. cofactor expansion). In connection to the research question, this shows that the pre-service teachers participating in the course have developed mainly procedural understanding while their conceptual understanding remains under construction.

This study may contribute to further research by showing how could the theoretical framework about procedural and conceptual understanding (Hiebert, 2013), in addition to the theory about MST (Leikin& Levav-Waynberg, 2007; Leikin, Levav-Waynberg, Gurevich, & Mednikov, 2006) and the theories about multiple modes of description and thinking (Hillel, 2000; Sierpinska, 2000) be used for analyzing students' achievements in undergraduate Linear algebra. In this article, these different theories are integrated through the three criteria for the solutions of MSTs in Linear algebra.

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NOTES

1. I come to the point of representations and properties of mathematical concepts in the section Multiple-Solution Tasks.

2. The complete task was: Find the determinants of the matrices:

$$A = \begin{pmatrix} 3 & 4 & 6 \\ 1 & -3 & 1 \\ 9 & 0 & -13 \end{pmatrix},\ B = \begin{pmatrix} 2 & 7 & 3 \\ -1 & 2 & 2 \\ 3 & 0 & -1 \end{pmatrix},\ C = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The total number of points was 12 and students' average score was 11.5 points.

3. Similar findings were derived, for example, by finding the determinant of C. There were students who used the Laplace expansion, exclusively for all four matrices A, B, C and D.

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