# A commognitive analysis of mathematics undergraduates' responses to a commutativity verification Group Theory task

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*The introduction of the notion of group is an important milestone in the study of mathematics at the university level. This study focuses on how students respond to this notion, especially during the process of verifying commutativity. Commognitive Theoretical Framework has been used in order to identify, from a participationist perspective, how students use the metalevel rules to prove the given task. Results suggest that although the overall response to the given task is satisfactory, there have been identified two categories of errors. The first category is related to the proof that commutativity holds, indicating problematic object level understanding of the particular axiom. The second category is related to the process of proof per se indicating inherited problematic engagement with metarules.*

*Keywords: Group Theory, Abelian Group, Commognition, Proof.*

## INTRODUCTION

Research focusing on the learning of Group Theory has attested on the university students' predicament to cope with this module, in their first engagement. Group Theory has proved to be a particularly demanding module for mathematics students, since they are required to successfully cope with its abstract and rigorous nature and invent new learning approaches. It is the first module in which students must go "beyond learning 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes (problems)" (Dubinsky et al, 1994, p268). Aim of this study is to elaborate further on the undergraduate mathematics students' reaction to the notion of abelian group, focusing in particular on its ontological characteristics, namely its structure and the axioms that should satisfy, i.e. associativity, existence of identity element, existence of inverses and, most particularly, commutativity. For the purposes of this study, there has been used the Commognitive Theoretical Framework (Sfard, 2008), due to its great potential to investigate learning in both object and meta-discursive levels (Presmeg, 2016).

## THEORETICAL FRAMEWORK

Commognitive Theoretical Framework (CTF) is a coherent and rigorous theory for thinking about thinking, grounded in classical Discourse Analysis (Yackel, 2009). It involves a number of different constructs such as metaphor, thinking, communication, and commognition, as a result of the link between interpersonal communication and cognitive processes, with commognition's five properties *reasoning*, *abstracting*, *objectifying*, *subjectifying* and *consciousness* (Sfard, 2008).

In mathematical discourse, unlike other scientific discourses, objects are discursive constructs and form part of the discourse. Mathematics is an *autopoietic system* of discourse, i.e. "a system that contains the objects of talk along with the talk itself and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p129). CTF defines discursive characteristics of mathematics as the *word use*, *visual mediators*, *narratives*, and *routines* with their associated metarules, namely the *how* and the *when* of the routine. In addition, it involves the various objects of mathematical discourse such as the *signifiers*, *realisation trees*, *realisations*, *primary objects* and *discursive objects*. It also involves the constructs of *object-level* and *metadiscursive level* (or metalevel) *rules*, along with their characteristics *variability*, *tacitness*, *normativeness*, *flexibility* and *contingency*.

Thinking "is an individualized version of (interpersonal) communicating" (Sfard, 2008, p81). Contrary to the acquisitionist approaches, participationists' ontological tenets propose to consider thinking as an act (not necessarily interpersonal) of communication, rather than a step primary to communication (Nardi et al. 2014; Sfard, 2012). Interpersonal communication processes and cognitive processes are (different) manifestations of the same phenomenon, and therefore Sfard (2008) combines the terms cognition and communication producing the new terms *commognition* and *commognitive*.

Sfard (2008) identifies the commognitive capacities that depend on the human ability to rise to higher commognitive levels and involve an "incessant interplay between utterances and utterances-on-former utterances" (Sfard, 2008, p110). These capacities fall into two distinct categories: those related to *commognitive objects* (i.e. reasoning, abstracting and objectifying), and those who consider the thinkers or speakers, namely the *commognitive subjects* (i.e. subjectifying and consciousness).

Mathematical discourse involves certain objects of different categories and characteristics. *Primary object* (p-object) is defined as "any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds)" (Sfard, 2008, p169). *Simple discursive objects* (simple d-objects) "arise in the process of proper naming (baptizing): assigning a noun or other noun-like symbolic artefact to a specific primary object. In this process, a pair <noun or pronoun, specific primary object> is created. The first element of the pair, the signifier, can now be used in communication about the other object in the pair, which counts as the signifier's only realization. *Compound discursive objects* (d-objects) arise by "according a noun or pronoun to extant objects, either discursive or primary." In the context of this study, groups are considered compound d-objects.

The (discursive) object signified by S in a given discourse is defined as "the realization tree of S within this discourse." (Sfard, 2008, p166) The *realization tree* is a “hierarchically organized set of all the realizations of the given signifier, together with the realizations of these realizations, as well as the realizations of these latter realizations and so forth” (Sfard, 2008, p300). Realisation trees and consequently mathematical objects are personal constructs, although they emerge from public discourses that support certain types of such trees. Additionally, realisation trees offer valuable information regarding the given individual’s discourse. Moving with dexterity from one realisation to another is the essence of mathematical problem solving. Realisation trees are a personal construction, which may be exceptionally ‘situated’ and easily influenced by external influences such as the interlocutors. Finally, signifiers can be realised by different interlocutors in different ways, according to their own specific needs (Sfard, 2008).

The epistemological tenet of CTF described in the last sentence is cardinal in its development as theoretical framework. Due to this tenet Sfard (2008) describes two distinct categories of learning, namely the *object-level* and the *metalevel discourse learning*. Moreover, according to Sfard (2008, p253), “object-level learning […] expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse”. In addition, “metalevel learning, which involves changes in metarules of the discourse […] is usually related to exogenous change in discourse. This change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses” (Sfard, 2008, p254).

CTF has proved particularly appropriate for the purposes of this study, since, as Presmeg (2016, p423) suggests, it is a theoretical framework of unrealised potential, designed to consider not only issues of teaching and learning of mathematics per se, but to investigate “the entire fabric of human development and what it means to be human.”

## LITERATURE REVIEW

Research in the learning of Group Theory is relatively scarce compared to other university mathematics fields, such as Calculus, Linear Algebra or Analysis. Even more limited is the commognitive analysis of conceptual and learning issues (Nardi et al. 2014). In the context of this research strand, Ioannou (2012) has, among other issues, focused on the intertwined nature of object-level and meta-level learning in Group Theory and the commognitive conflicts that emerge.

The first reports on the learning of Group Theory appeared in the early 1990’s. Several studies, following mostly a constructivist approach, and within the Piagetian tradition of studying the cognitive processes, examined students’ cognitive development and analysed the emerging difficulties in the process of learning certain group-theoretic concepts.

The construction of the newly introduced d-object of group is often an arduous task for novice students and causes serious difficulties in the transition from the informal secondary education mathematics to the formalism of undergraduate mathematics (Nardi, 2000). Students' difficulty with the construction of the Group Theory concepts is partly grounded on historical and epistemological factors: "the problems from which these concepts arose in an essential manner are not accessible to students who are beginning to study (expected to understand) the concepts today" (Robert and Schwarzenberger, 1991). Nowadays, the presentation of the 'fundamental concepts' of Group Theory, namely group, subgroup, coset, quotient group, etc. is "historically decontextualized" (Nardi, 2000, p169), since historically the fundamental concepts of Group Theory were permutation and symmetry (Carspecken, 1996). Moreover, this chasm of ontological and historical development proves to be of significant importance in the metalevel development of the group-theoretic discourse for novice students.

From a more participationist perspective, CTF can prove an appropriate and valuable tool in our understanding the learning of Group Theory due both to the ontological characteristics of Group Theory, as well as the epistemological tenets of CTF. Group Theory can be considered as a metalevel development of the theory of permutations and symmetries. Moreover, CTF allows us to consider the historical and ontological development of a rather 'historically decontextualized' modern presentation of this Theory.

Research suggests that students' understanding of the d-object of group proves often primitive at the beginning, predominantly based on their conception of a set. An important step in the development of the understanding of the concept of group is when the student "singles out the binary operation and focuses on its function aspect" (Dubinsky et al, 1994, p292). Students often have the tendency to consider group as a 'special set', ignoring the role of binary operation. Iannone and Nardi (2002) suggest that this conceptualisation of group has two implications: the students' occasional disregard for checking associativity and their neglect of the inner structure of a group. These last conclusions were based on students' encounter with groups presented in the form of group tables. In fact, students when using group tables adopt various methods for reducing the level of abstraction, by retreating to familiar mathematical structure, by using canonical procedure, and by adopting a local perspective (Hazzan, 2001).

An often-occurring confusion amongst novice students is related to the order of the group G and the order of its element g. This is partly based on student inexperience, their problematic perception of the symbolisation used and of the group operation. The use of semantic abbreviations and symbolisation can be particularly problematic at the beginning of their study. Nardi (2000) suggests that there are both linguistic and conceptual interpretations of students' difficulty with the notion of order of an element of the group. The role of symbolisation is particularly important in the learning of Group Theory, and problematic conception of the symbols used probably causes confusion in other instances.

A distinctive characteristic of advanced mathematics in the university level is the production of rigorous and consistent proofs. Proof production is far from a straightforward task to analyse and identify the difficulties students face. These difficulties have been extensively investigated for various levels of student expertise. Weber (2001) categorises student difficulties with proofs into two classes: the first is related to the students' difficulty to have an accurate and clear conception of what comprises a mathematical proof, and the second is related to students' difficulty to understand a mathematical proposition or a concept and therefore systematically misuse it.

## METHODOLOGY

This study is part of a larger research project, which conducted a close examination of Year 2 mathematics students' conceptual difficulties and the emerging learning and communicational aspects in their first encounter with Group Theory. The module was taught in a research-intensive mathematics department in the United Kingdom, in the spring semester of a recent academic year.

The Abstract Algebra (Group Theory and Ring Theory) module was mandatory for Year 2 mathematics undergraduate students, and a total of 78 students attended it. The module was spread over 10 weeks, with 20 one-hour lectures and three cycles of seminars in weeks 3, 6 and 10 of the semester. The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, and the sessions were each facilitated by a seminar leader, a full-time faculty member of the school, and a seminar assistant, who was a doctorate student in the mathematics department. All members of the teaching team were pure mathematicians.

The lectures consisted largely of exposition by the lecturer, a very experienced pure mathematician, and there was not much interaction between the lecturer and the students. During the lecture wrote self-contained notes on the blackboard, while commenting orally at the same time. Usually, he wrote on the blackboard without looking at his handwritten notes. In the seminars, the students were supposed to work on problem sheets, which were usually distributed to the students a week before the seminars. The students had the opportunity to ask the seminar leaders and assistants about anything they had a problem with and to receive help. The module assessment was predominantly exam-based (80%). In addition, the students had to hand in a threefold piece of coursework (20%) by the end of the semester.

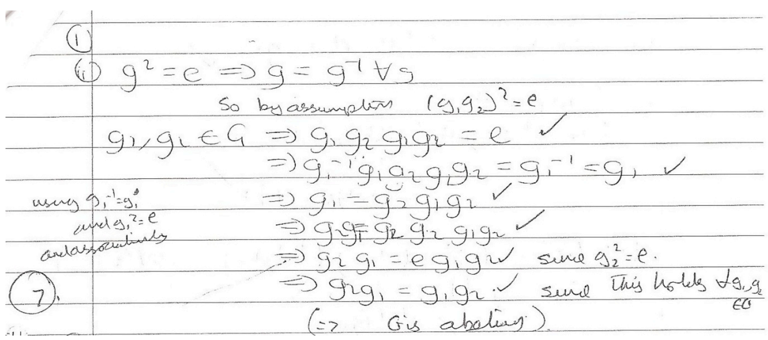
The gathered data includes the following: Lecture observation field notes, lecture notes (notes of the lecturer as given on the blackboard), audio-recordings of the 20 lectures, audio-recordings of the 21 seminars, 39 student interviews (13 volunteers who gave 3 interviews each), 15 members of staff's interviews (5 members of staff, namely the lecturer, two seminar leaders and two seminar assistants, who gave 3 interviews each), student coursework, markers' comments on student coursework, and student examination scripts. For the purposes of this study, the collected data of the 13 volunteers has been scrutinised. Finally, all emerging ethical issues during the data collection and analysis, namely, issues of power, equal opportunities for participation, right to withdraw, procedures of complain, confidentiality, anonymity, participant consent, sensitive issues in interviews, etc., have been addressed accordingly.

## DATA ANALYSIS

In the first piece of coursework, students needed to prove the following task, which was solely focused on the ontology of the concept of group: *Suppose $(G, \circ)$ is a group with the property that* $g^2 = e$, for all $g \in G$. Prove that for all $g_1, g_2 \in G$, we have $g_1 g_2 = g_2 g_1$ *(that is, G is abelian).*

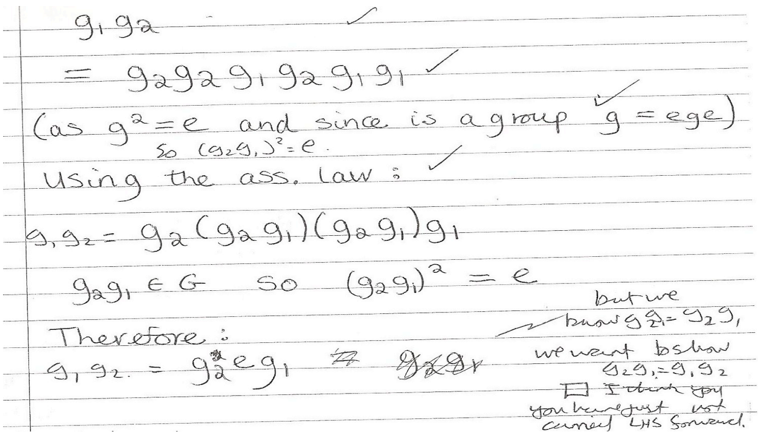
In general, students' encounter with the d-object of group was satisfactory, and their performance was generally in agreement with their impression as this has been revealed in their interviews. Their understanding was generally quite explicit and their mathematical narratives show good use of the mathematical vocabulary and notation, as well as the ability to specifically demonstrate their reasoning in the specific routine. Yet there have occurred **two types of errors** concerning the proof that the group is Abelian: errors in the process of *proving that the group is abelian*; and assumption of *what needs to be proved*;

***Problematic proof that a group is Abelian*** was mostly related to the use of group axioms, and in particular commutativity. These inaccuracies are possibly linked with the incomplete object-level understanding of the definition of group and the involved object-level rules. As the excerpt below suggests, student A assumes that $(g_1 g_2)^2 = e$, but she rather takes it for granted. As seen below, she applies all the necessary manipulations of $g \in G$, for instance $g_1{}^{1} = g_1$ and $g_1^2 = e$ as well as associativity, but with no further explanation. In addition, she does not clearly state that since $g_1 g_2 = g_2 g_1$ therefore $G$ is Abelian. This indicates an incomplete object-level understanding of the property of commutativity, or an inaccurate application of the governing metarules, resulting deficient presentation of her reasoning.



**Excerpt 1: Solution of Student A**

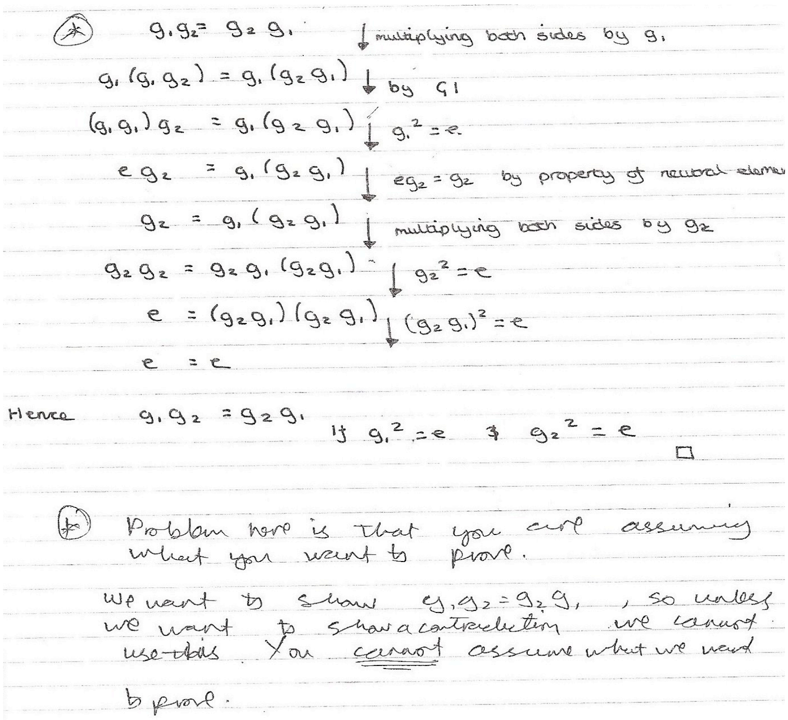
Another inaccuracy is related to the actual proof of the expression $g_2g_1 = g_1g_2$. Student B, as the excerpt below indicates, shows good object-level understanding of the definition of group, and demonstrates the ability to use the group axioms and apply the object-level rules for proving that the group is Abelian, yet he does not always justify his steps. He correctly multiplies both sides of the expression $g_1g_2$ by $g_2^2$ and $g_1^2$ respectively and correctly uses associativity. The main problem with his solution appears at the end of the exercise, where a problematic understanding of the notion of commutativity becomes apparent. Instead of completely demonstrating that $g_1g_2 = g_2g_1$, he erases the second part $g_2g_1$, which puts in doubt his solutions' endeavour as well as his understanding of the definition of Abelian group. This error indicates problematic application of the object-level rules related to commutativity and the manipulation of the group elements.



**Excerpt 2: Solution of Student B**

A second type of error was grounded on the assumption of ***what was supposed to be proved***. This error was usually part of an overall satisfactory attempt that would suggest an explicit understanding of the object-level rules of the d-object of group, but would highlight a problematic encounter with the metalevel rules and the 'how' of proving, even during the very first step of the module. This kind of errors is a typical misapplication of the metalevel rules, since it is directly linked with the 'norms' of proving and not of the d-objects as such.

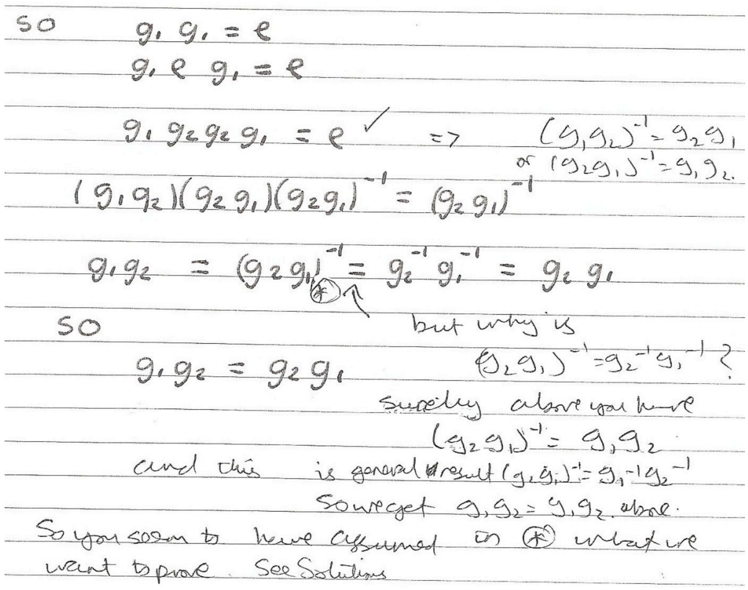
Problematic application of metarules does not require problematic object-level understanding of the d-objects under study. In the following excerpt Student C's writing style is very analytical with very clear mathematical narratives, good presentation and explicit use of symbolisation, in all her written mathematical narratives. Although a high performer, in this task she assumed what was supposed to be proved at the beginning of the solution i.e. $g_1g_2 = g_2g_1$. This indicates an unawareness of how to approach a proof of this kind and the required course of action, and consequently leads to a problematic encounter with this type of routine and the amenable metarules. In general, a significant obstacle in the application of metarules, as the following excerpt suggests, is the distinction between the different proving techniques and how the amenable metarules should be used. For instance, assuming that a certain mathematical narrative is valid and used within the proof is only applied in proof by contradiction, which is not the case in this exercise.




**Excerpt 3: Solution of Student C**

Another example of using what is supposed to be proved within the proof occurred in Student D's solution. He assumes that what he is trying to prove is valid and uses it during the proof, which indicates problems in applying a fundamental metadiscursive rule regarding the role of the 'to-be-proved' mathematical narrative. As in the other two cases, this is possibly directly linked with the metalevel understanding and indicates ignorance of the governing metarules. At the same time, the step indicated by *, reveals an object-level error regarding the manipulation of group theoretical expression $(g_2g_1)^{-1}$. Although he has proven that $(g_2g_1)^{-1} = g_1g_2$, at a later stage, instead of writing $(g_2g_1)^{-1} = g_1^{-1}g_2^{-1}$ he has written $(g_2g_1)^{-1} = g_2^{-1}g_1^{-1}$, which indicates problematic object-level understanding of the notion of inverse. The following interview excerpt reinforces the last assumption.

> Yeah, I just weren't sure whether I'd done it right, whether I was allowed to do it that way, cos I think I – I did something with like timesing it by, like both of them and then timesing $g_1$ them by $g_2$ and getting like the identity and stuff, but – I weren't sure whether it was the right way to do it? But I came out with like, the answer, but...



**Excerpt 4: Part of the solution of Student D**

The fact that some students use mathematical narrative that needs to be proved, as a datum that should be used during the proof, possibly indicates a problematic engagement with the 'how' of the routine. Namely, they have not a stabilised strategy about the 'course of action' for the given mathematical task. These particular students, probably, have not yet a clear idea of how to approach a proof, what the role of the final statement is and how to use it in order to achieve rigorous and clear proof. Consequently, the closure conditions, the set of metarules that define circumstances interpreted as signalling a successful completion of the proof, are not clear to these students. This is suggested by the fact that these students are generally not aware that the final narrative should not be used in the middle of the proof, albeit the fact that in the interviews students occasionally express their uncertainty regarding their solution.

## CONCLUSION

This study's aim was to investigate from a commognitive perspective undergraduate mathematics students' reaction to the notion of Abelian group, focusing in particular on its ontological characteristics, namely its structure and its axioms. As the above analysis suggests, there have emerged two types of errors. The first one is related to the proof that commutativity holds and therefore the group is Abelian. This error is a result of problematic object-level understanding of the axiom of commutativity. The second error occurred when students used the mathematical statement that was supposed to be proved during proof. This error is related to problematic metalevel learning and application of the metarules that govern the particular routine, causing of a commognitive conflict. Moreover, the above discussion indicates that, at the early stages of the module, students' object-level understanding is better compared to the metalevel understanding and the application of the required metarules. For some students, metalevel understanding appears to be problematic from the very beginning.

## REFERENCES


Carspecken, P. F. (1996). *Critical Ethnography to Educational Research*. London: Routledge.

Dubinsky, E., Dautermann J., Leron U., Zazkis R. (1994). On learning the fundamental concepts of Group Theory. *Educational Studies in Mathematics 27*, 267-305.

Hazzan, O. (2001). Reducing abstraction: The case of constructing an operation table for a group. *Journal of Mathematical Behavior 20*, 163-172.

Iannone, P. & Nardi, E. (2002). A group as a special set? Implications of ignoring the role of the binary operation in the definition of a group. *In Proceedings of 26th Conference of the International Group for the Psychology in Mathematics Education*. Norwich, UK.

Ioannou, M. (2012). *Conceptual and learning issues in mathematics undergraduates' first encounter with group theory: A commognitive analysis* (Unpublished doctoral dissertation). University of East Anglia, UK.

Nardi, E. (2000). Mathematics undergraduates' responses to semantic abbreviations, 'geometric' images and multi-level abstractions in Group Theory. *Educational Studies in Mathematics 43*, 169-189.

Nardi, E., Ryve A., Stadler E., Viirman O. (2014). Commognitive Analyses of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of Calculus. *Research in Mathematic Education 16,* 182-198.

Presmeg, N. (2016). Commognition as a lens for research. *Educational Studies in Mathematics 91,* 423-430.

Robert, A. & Schwarzenberger, R. (1991). Research in teaching and learning mathematics at an advanced lever. D. Tall. (Ed.) *Advanced Mathematical Thinking* (127-139). Dordrecht, Boston, London: Kluwer Academic Publishers.

Sfard, A. (2008). *Thinking as Communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Sfard, A. (2012). Introduction: Developing mathematical discourse – Some insights from communicational research [Editorial]. *International Journal of Educational Research 51-52,* 1-9.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics 48*, 101-119.

Yackel, E. (2009). Book Review: Thinking as communicating: human development, the growth of discourses, and mathematizing. *Research in Mathematics Education 11*, 90-94.