Designation at the core of the dialectic between experimentation and proving: a study in number theory

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It is well known in mathematics education that students feel strong difficulties for elaborating by themselves mathematical proofs, even when they are involved in meaningful solving research problem activities. There are research based evidences that even in settings where the milieu for validation seems to be rich enough to support the proving process, some students fail to enter appropriately into it. In this paper, we provide some empirical results supporting the following hypothesis: although the designation of objects plays an important role in the heuristic phases, it might not be sufficient to enrol students in elaborating proof in cases the properties of these objects and their mutual relationships are not made explicit.

Keywords: solving research activities, proof and proving skills, gesture, designating an object

INTRODUCTION

It is well known in mathematics education that students feel strong difficulties for elaborating by themselves mathematical proofs. At first glance, one might think that a relevant way to improve proof and proving skills is to involve students in meaningful solving research problem activities (Durand-Guerrier & al., 2012). However, there are research based evidences that even in settings where the *milieu* for validation (Brousseau, 1997) seems to be rich enough to support the proving process, some students fail to enter appropriately into it (e.g. Tanguay & Grenier, 2010). Relying on an epistemological study, we acknowledge that *designation* plays an important role in the proof and proving process, including the heuristic phases. In this paper, we provide some empirical results supporting the following hypothesis: although the *designation of objects* plays an important role in the heuristic phases, it might not be sufficient to get students enrolled in elaborating proof in instances where the properties of the involved objects and their mutual relationships are not made explicit. In the first section, we present a didactical engineering (González-Martín & al., 2014) in Number Theory from which the empirical data were collected, and aiming at fostering the development of students' skills for solving mathematical research problems. In the second section, we present the concept of *gesture* adapted from Philosophy of Mathematics. Introduced by Cavaillès (1981), it has been developed in particular by Châtelet (1993) and more recently by Longo (2005), in order to analyse mathematical activity. We first briefly present seven gestures that appear to be relevant for analysing the research process carried out by researchers and students, and then we develop more on the *gesture* “designating an object”. In the third section, we analyse some empirical data out of Gardes (2013) in order to support the above mentioned hypothesis.

A DIDACTICAL ENGINEERING IN NUMBER THEORY

The didactical engineering that we present below is part of a research project (Gardes, 2013) whose main goal was to study the conditions and constraints pertaining to the transposition of professional mathematicians’ research activity in the mathematical classroom. The didactical engineering has been elaborated from an unsolved problem in Number Theory, the Erdös-Straus conjecture: “the equation $\frac{4}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ can be solved in positive integers x, y and z for any integers $n > 1$”. The general methodology of the research consists in three interrelated studies in mathematics, epistemology and didactics. The mathematical and epistemological analyses included observation, running over three years, of the work from two mathematicians on this problem. This contemporaneous epistemological study along with a more classical study on the context of discovery in mathematics (Gardes, 2013, pp. 108-149) allow the identification of various strategies for entering into and progressing through the research process: *reducing the problem to prime numbers, transforming the original equation while preserving equivalence, constructing effective decomposition, constructing and implementing algorithms.* Relying on Cavaillès (1981) and Châtelet (1993), Gardes (2013) adapted the concept of *gesture* in mathematics activity in order to describe, analyse and contextualize the research processes that appeared in the various situations that have been experimented around the Erdös-Strauss conjecture in the frame of the research project. This is developed below in section 2.

Several pre-experimentations at different levels (middle school (grade 7 and 9); high school (grade 12) and university (first and third-year)) have been carried out in order to test the suitability of this situation for secondary and tertiary students, and to determine the main characteristic of a *milieu* (Brousseau, 1997) favouring the involvement of students in a genuine and rich problem solving activity on the Erdös-Straus conjecture (the *devolution*, Brousseau, 1997). We have identified the following core elements: 1. For the students: the availability of mathematical and heuristics knowledge and a frequent practice of mathematical research activities; 2. For the didactical engineering itself: several research sessions with a very precise organisation, a formulation of the conjecture involving verbs of action, and availability of programmable calculators or computers. These elements have been taken into consideration when elaborating the main experimentation, in which we tried to control as precisely as possible the didactical conditions and constraints that we had identified. This experimentation took place in spring 2012 during two months with ten students from grade 12 (17 years old) who followed a two-hour-per-week specific “excellence program”, aiming at preparing them for University, in addition to a two-year optional course providing, among other things, a rich mathematical background in Number Theory: *familiarity with congruence and algorithms; Bezout theorem, Gauss theorem, fundamental theorem of arithmetic,* etc. This experimentation consisted of seven two-hour sessions: one individual research session; four collective sessions devoted to research in small groups (three groups of three or four students); one debate session and one synthesis session.

During the specific weekly sessions of the "excellence program", the students had been regularly engaged in research activities aiming at developing heuristics, such as recognising the epistemic status of a conjecture and the role of examples and counterexamples in the development of proof and proving; availability of different types of arguments and modes of reasoning; recognising the importance of taking into consideration intermediate and partial results. The students were able to apprehend different aspects of mathematical research: learning from the unsuccessful phases, identifying the diversity of approaches, being able to use various frameworks and to make links between different fields, etc. We hypothesised that, through these didactic experiences, these students elaborated a rather adequate representation of mathematical activity in general. In addition, the core elements mentioned above have been taken into account in the construction of the didactical engineering, so that the initial *milieu* included all the elements that we identified in order to favour the *devolution* of the problem over a long period of time and to foster significant advances and fruitful research developments.

In the next section, we present the concept of *gesture* that we have developed and we examine in particular the fundamental gesture consisting in "*designating an object"*.

THE CONCEPT OF *GESTURE* IN OUR RESEARCH

The concept of *gesture* comes from the philosophy of mathematics (Cavaillès, 1981, 1994; Châtelet, 1993, Bailly & Longo, 2003, Longo 2005). It allows considering different aspects of the mathematician's work: active dimension of research, central role of intuition in the creative process and dialectical aspects between acquisition of knowledge and development of heuristics, and of skills in proof and proving. In our didactic perspective, we consider the following definition, adapted from Cavaillès and Châtelet:

> A *gesture* is an action connecting mathematical objects and which is carried out with intentionality. It is an operation that is accomplished through a combination of signs with respect to the usage rules of these signs. Because they open on possibilities, *gestures* have the power to enhance mathematical creativity (translated from Gardes, 2013, p. 155).

Relying on our epistemological study (both historical and contemporaneous) and on our pre-experimentations, we have identified seven gestures that appear to be relevant for analysing the research process carried out by researchers and students:

- *Reducing the problem to prime numbers*: this is relevant in the Erdös-Straus problem due to the fact that the property is multiplicative;
- *Designating an object* i.e., representing a mathematical object by means of a natural language expression or a symbol;

- *Introducing a parameter in a mathematical writing*: this allows making visible some relationships between two or more involved objects, without these objects being explictly referred to;
- *Constructing and questioning examples*: to frame a method of constructing examples from manipulation of mathematical objects, and to study these different examples in order to generate information;
- *Making local controls*: checking the different stages of manipulation and combinations of signs in mathematical writings;
- *Transforming the original equation while preserving equivalence*;
- *Implementing an algorithm*: translating a mathematical algorithm in a programming language.

In this paper, we examine specifically the *gesture* "designating an object" by means of a language expression or a symbol, a gesture we consider as crucial for heuristics. In particular, when an object is designated by a symbol, it is possible to perform operations involving this symbol, temporarily leaving aside the reference. In our experiments, in some cases this *gesture* has supported theoretical development. In other cases, an adequate designation of an object allowed a reformulation of the conjecture entailing an enrichment of the initial problem. More generally, we observed in many cases that this *gesture* enabled to keep advantage of the experimental aspects of the problem, fostering the back-and-forth between manipulation of objects and theoretical elaborations, or in other words the dialectics between semantics and syntax (Gardes, 2012).

During the heuristic phases of the students' work, the *gesture* "designating an object" emerged from the manipulation of the involved objects (fractions, integers) in interrelation with the mathematical knowledge at stake. So, it played an important role in the students' research at several levels: to introduce some mathematical objects they used as tools to advance in research (prime numbers, congruence); to provide intermediate conjectures and to have the various steps of their method written down and formalized. Nevertheless, our *a posteriori* analyses of the students' work (in particular from Group 3) show that it might be insufficient for getting enrolled in a proving process, as we will see in section 3. We hypothesise that in order to foster the involvement in proving, the *gesture* "designating an object" should make visible the properties of numbers and the relationship between the numbers involved.
In the next section, we specifically examine the phases of proof construction in the students' work of two groups (group 1 and group 3) to support this hypothesis.

DESIGNATION IN STUDENTS' WORK

During the experimental part of their work, the students in group 3 have obtained many results, mainly a method of decomposition of $\frac{4}{n}$ for a given value of n (n a prime number), including six identities and the verification of the conjecture of Erdös-Straus for $0 < n < 300$ (Gardes, 2013, pp. 445-449). However, although the group members explicitly recognize the necessity of a proof, they meet resisting difficulties to engage themselves into a proving process relying on their experimental results. This is in accordance with results from Tanguay and Grenier (2010) concerning the relationships between activities of definition, of construction and of proving in Geometry. We conjecture that the main obstacles to their involvement in the proving process pertain to the choices the students made in order to designate the mathematical objects, choices that do not enlighten the properties of these objects and their mutual relationships, opposite to what appeared in group 1.

We present below elements of the *a posteriori* analysis (Gardes, 2013, pp. 315-475) that support this claim. We will focus on three phases of the didactical engineering: the session 3 in which students share their results and methods; the session 4 devoted to students work in their own small group and the synthesis of group 1 and 3 work, written at the request of the researcher at the end of session 4.

Session 3 - Students share the first results obtained within the three groups

During the third session, the teacher organised exchanges between the students: in each group, students had prepared a synthesis of their work that was being presented to their classmates. Each presentation was followed by a reaction from students of the other groups. Below are some excerpts of the presentation of group 3, and some students' reaction from groups 1 and 2.
Students in group 3 are presenting a method for providing a decomposition of the fraction $\frac{4}{n}$ as a sum of two fractions: $\frac{4}{n}=\frac{1}{y}+\frac{x}{z}$, where *x*, *y* et *z* are non zero natural numbers. They explain that they are looking for the greatest value of $\frac{1}{y}$ satisfying the property "$\frac{4}{n}-\frac{1}{y}$ is positive". This value if found by essay-errors with the calculator ie. students were trailing different numbers in the calculators. Then they try to decompose the fraction $\frac{x}{z}$ as a sum of two fractions with numerator 1 (Egyptian fractions). The other students ask several questions; they try to understand how to find the value of y.

> Student (group 3): we take a prime number by chance, for example $\frac{4}{457}$ and we try to find a fraction such that $\frac{4}{457}$ minus for example $\frac{1}{115}$ is positive; if we take $\frac{1}{114}$ it becomes negative. [...] We know that it is the smallest natural number.
> Student (group 2): $\frac{1}{115}$, did you find it with the calculator, making test?
> Student (group 3): Feeling blindly, we did not manage to do it with greater numbers.

A student in group 2 proposes to consider the integer part of the number $\frac{n}{4}$ in order to determine the closest Egyptian fraction. A student in group 3 replies:

> Student (group 3): it would be more logic, we should make tests.

At this point, it seems to the researcher that the explicit designation of the integer part would allow to put forward the method presented by the students of group 2 and to support an involvement in proof and proving. As will be shown below, this is actually the case only for the students in group 1. If we analyse this with the concept of the situation *milieu*, we can say that for the students in group 2 and 3, although this came after the discussion, the designation was in the *materiel milieu* of the students but for some of them, it was not in their *heuristic milieu* (González-Martín & al., 2014).

Session 4 - Students came back working in their respective groups

Following the discussion, students in group 3 wrote $\frac{4}{n} = \frac{1}{t} + \frac{j}{k}$ where *t* is the smallest natural number such that $\frac{4}{n} - \frac{1}{t} > 0$, and *j* and *k* are non zero natural numbers. This equation allowed them to find relevant values for *t* by essays-errors using the calculator, and so to determine several decompositions of $\frac{4}{n}$ for given values of *n*. This equation with four variables (*n, t, j* and *k*) is efficient for providing decomposition; however, it does not show possible relationships between some of these variables, in particular the relationship between *n* and *t* that had been explicitly stated during the collective discussion through the denotation of the integer part of the number $\frac{n}{4}$.

Opposite, students in group 1 developed this idea of an explicit relationship between *n* and *t* through the use of the integer part, and tried to generalize it. To that purpose, they thought of using the notation $E\left(\frac{n}{4}\right) + 1$. They then wrote $\frac{4}{n} = \frac{1}{E\left(\frac{n}{4}\right)+1} + \frac{x}{n\times\left(E\left(\frac{n}{4}\right)+1\right)}$, with x a non-zero natural number. This allowed them to determine the value of *t* for given values of *n*.

Seeing the decompositions provided by both groups 1 and 3 (annex 1, annex 2), one might get the impression that the two equations have the same efficiency. The difference appears when we consider the involvement in proving.

Comparing the syntheses of groups 1 and 3

After having written their equation $\frac{4}{n} = \frac{1}{E\left(\frac{n}{4}\right)+1} + \frac{x}{n\times\left(E\left(\frac{n}{4}\right)+1\right)}$, students in group 1 tried to determine a decomposition of the second fraction ($\frac{x}{n\times\left(E\left(\frac{n}{4}\right)+1\right)}$) as the sum of two Egyptian fractions. Relying on examples (*n* = 29, *n* = 457, *n* = 461, *n* = 4513), they conjecture that $if\ n \equiv 3[4]\ then\ x = 1$ and $if\ n \equiv 1[4]\ then\ x = 3$. As they had already a decomposition of $\frac{4}{n}$ in sum of three Egyptian fractions when *n* is even, they recognized that with these results, they were covering all cases pertaining to the determination of *y*. The designation of *y* by $E\left(\frac{n}{4}\right) + 1$ allowed these students to getting successfully engaged in the proving process, reaching an important intermediate result. Then, they considered the case where $x = 1$ and managed to provide the following general decomposition for $n \equiv 3[4]$: $\frac{4}{n} = \frac{1}{y} + \frac{1}{2z} + \frac{1}{2z}$. They then considered the case $x = 3$, and made a reasoning by distinction of cases, considering the values of *z* modulo

6: they established a general decomposition in the cases $z \equiv 0[6], z \equiv 2[6]$ and $z \equiv 3[6]$. They failed to find a general decomposition for the two remaining cases[1].

In group 3, students first wrote their equation: $\frac{4}{n} = \frac{1}{t} + \frac{j}{k}$, where *t* is the smallest natural number such that $\frac{4}{n} - \frac{1}{t} > 0$, and j and k are non-zero natural numbers. They distinguished cases that they studied successively. The first case they studied is $j = 1$; they got the equation $\frac{4}{n} = \frac{1}{t} + \frac{1}{2k} + \frac{1}{2k}$; then they studied the case $j = 3$ and *k* even, and got the new equation: $\frac{4}{n} = \frac{1}{t} + \frac{1}{k} + \frac{2}{k}$. The third case they studied is $j = 3$ and *k* odd, with as a first attempt the case where *k* is a multiple of 5.

According to us, the comparison of the synthesis of these two groups enlightens the role played by the choice of designation made by students in order to transform their equations. In the first group, the designation of *y* by the symbolic notation $E\left(\frac{n}{4}\right) + 1$ allowed them to enter into fruitful transformations of their initial equation, opening the possibilities of reasoning by disjunction of cases and so getting stabilized results: identification of classes of natural numbers for which we have a general formula providing a decomposition – identification of the remaining cases that needed to be continued in the research. In the third group, although students showed cleverness in finding operative patterns to perform decompositions, they did not manage to determine the classes of natural numbers for which the answer is positive, and those classes for which the question remains open.

CONCLUSION

The *a posteriori* analyses of the work of these two groups show that for both groups, the designation gesture promotes involvement in the proving process. Nevertheless, the work of the third group showed that for these students, although their gestures of designation of objects are fruitful in the heuristic phases, they met difficulties to enter in general proof. At the opposite, the choices of designation made by the students of the first group, by showing explicitly the relationship between *n* and *y*, allowed them to establish general intermediate results and to identify the cases that need further studies. This supports our hypothesis that the *gesture* of "designation of object" might not be sufficient to enter successfully in proof and proving when this designation does not make explicit the properties and the involved relationships. This is a point that we should take into account when elaborating the *milieu* of such didactical engineering, aiming at fostering the development of students' skills for solving mathematical research problems. The way to do this is an open research question.

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[1] Currently, the numbers for which the Erdos-Strauss equation does not hold is a part of one of these classes. For details on a synthesis of mathematical results: Gardes (2013, pp.73-103).

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ANNEXE 1: STUDENTS' WORK (GROUP 1) – SYNTHESIS[2]

We use the following wrinting:

$$\frac{4}{n} = \frac{1}{y} + \frac{x}{z}$$

If we put $y = ent\left(\frac{n}{4}\right) + 1$, we have:

$$\frac{4}{n} = \frac{1}{ent\left(\frac{n}{4}\right) + 1} + \frac{x}{n \times \left(ent\left(\frac{n}{4}\right) + 1\right)}$$

So, we try to write $\frac{x}{n \times \left(ent\left(\frac{n}{4}\right)+1\right)}$ as a sum of two fractions with numerator 1 ie. with the form $\frac{1}{k}$ → $x = 1$ or $x = 3$ (even if *n* is even and therefore not prime number)

- If $x = 1$ then $\frac{4}{n} = \frac{1}{ent\left(\frac{n}{4}\right)+1} + \frac{1}{2n \times \left(ent\left(\frac{n}{4}\right)+1\right)} + \frac{1}{2n \times \left(ent\left(\frac{n}{4}\right)+1\right)}$
- If $x = 3$, we reason modulo 6.

If $z \equiv 0[6]$ or $z \equiv 3[6]$, $\frac{3}{z} = \frac{3}{3k} = \frac{1}{k} = \frac{1}{2k} + \frac{1}{2k}$.

Then $\frac{4}{n} = \frac{1}{y} + \frac{1}{2k} + \frac{1}{2k}$

If $z \equiv 2[6]$ or $z \equiv 4[6]$ → z is divisible by 2, then $z = 2k$.

$$\frac{3}{z} = \frac{1}{z} + \frac{2}{z} = \frac{1}{2k} + \frac{1}{k}$$

Then $\frac{4}{n} = \frac{1}{y} + \frac{1}{2k} + \frac{1}{k}$.

If $z \equiv 1[6]$ or $z \equiv 3[6]$, we have not found any possibility to make a decomposition of $\frac{3}{z}$ as a sum of two fractions with numerator 1 (ie. with the form $\frac{1}{k}$).

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[2] Translated by the authors. The original work is in (Gardes, 2013, Annexes, pp.83-84).

ANNEXE 2: STUDENTS' WORK (GROUP 3) – SYNTHESIS[3]

- If *n* is even $\frac{4}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{\frac{n}{2}}$ (with *n* even)
- If *n* is odd and prime
  ➔ $\frac{4}{n} = \frac{1}{t} + \frac{j}{k}$ with t the smallest natural number and j, k non-zero natural numbers.

There are 3 cases.

- <u>Cas 1</u> :
  $j = 1$ with $\frac{j}{k}$ an irreducible fraction.
  $$\frac{4}{n} = \frac{1}{t} + \frac{1}{2k} + \frac{1}{2k}$$

- <u>Cas 2</u> :
  $j = 3$ with k even.
  $$\frac{4}{n} = \frac{1}{t} + \frac{1}{k} + \frac{2}{k}$$

- <u>Cas 3</u> :
  - $j = 3$ and *k* is a multiple of 5. Thus $\frac{4}{n} = \frac{1}{t} + \frac{1}{2k} + \frac{5}{2k}$.
  - $j = 3$ and *k* is odd
    $\frac{4}{n} = \frac{1}{t} + \frac{3}{k}$. The number *k* is decomposed into prime factors.

    <u>Subcase 1</u>:
    There is at least one of divider congruent to 2 modulo 3 such as $k = d \times q$
    For this case, we note that $\frac{4}{n} = \frac{1}{t} + \frac{3}{n \times t}$.
    Put $\frac{d+1}{3} = e$ with e even.
    We multiply $\frac{3}{nt}$ and $\frac{e}{e}$

[3] Translated by the authors. The original work is in (Gardes, 2013, Annexes, pp.196-197).