Multiple choice questions and peer instruction as pedagogical tools to learn the mathematical language

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Previous research has shown that the use of clicker questions and Peer Instruction in a lecture can have a positive impact on students' understanding, especially their conceptual understanding. The quality of students' discussions plays a crucial role for increasing the understanding. However, little is known about the role that clicker questions play in triggering high quality collaborative discussions in undergraduate analysis courses. In this case study, I will show how a clicker question, designed to help understand AE and EA expressions, triggered the meaning making process of one group. Different interpretations of the expression were an ideal trigger to a high quality discussion. At the end I set up some hypotheses about the design of good clicker questions.

INTRODUCTION

At the beginning of their studies students have to face many challenges. One major problem that math students face is to learn the mathematical language. The modern symbolic mathematical language developed over centuries and became increasingly dominant from early 19th century onwards (Nardi, 2011, p. 2053). This language is subject to rules that are beyond the rules of ordinary language (Schichl & Steinbauer, 2009, p. 8). These rules have to be learned quickly, because most of the mathematical content - especially in lectures - is presented in this language. Therefore lecturers should ask themselves, how they can support the process of learning this language. One possibility is to integrate multiple choice questions (clicker questions[1]) with Peer Instruction (PI) into lectures themselves, as recommended by Mazur (1997): the lecturer presents a clicker question, the students vote for the first time, discuss their vote for a few minutes with their neighbours (this is PI) and then re-vote a second time before the solution and reasoning is explained. Numerous research studies showed that this method can increase conceptual understanding(e.g. Deslauriers, Schelew, & Wieman, 2011; Freeman et al., 2014; Hake, 1998). Moreover Smith et al. (M. K. Smith et al., 2009; M.K. Smith, Wood, Krauter, & Knight, 2011) showed the particular importance of PI for this increase.

Clicker questions for learning the mathematical language

Increasing conceptual understanding alone does not guarantee a better understanding of the rules of mathematical language and of typical mathematical expressions. But there are two reasons why PI can support the understanding. It allows students to deal with course material on their level of understanding and they have to express their

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[1] *Multiple Choice question like the presented one here were often named clicker question because many lecturers use clicker devices in their lectures.*

ideas in ordinary language. This kind of verbalisation and reasoning can help to understand mathematical expressions. It “can act as a crucial semiotic mediator between symbolic and visual mathematical expression” (Nardi, 2011, p. 2060).

Research Question

So collaboration during PI is important to improve the understanding. A mathematical discussion becomes collaborative when it is useful for the task at hand, and the students “explore each other`s reasoning and viewpoints while working on a common activity, so that shared understanding evolves simultaneously for all participants” (Goos, Galbright, & Renshaw, 1996, p. 237).

A study (Knight, Wise, & Southard, 2013) that investigated the quality of collaboration during PI found a correlation between the quality of the clicker question and the quality of students’ discussion. In this paper I will analyse the role of clicker questions in triggering the discussion and the construction of new mathematical knowledge of the group. This deep insight might help create high quality clicker questions, especially to support students’ in learning the language of mathematics.

Construction of mathematical knowledge through interaction

The idea of PI is to allow students tocreate new mathematical knowledge by discussing with their neighbours. Creating new mathematical knowledge cannot be seen as a given product to which further knowledge elements can simply be added. Instead it has to be understood as an extension of the old knowledge by means of new, extensive relations and allow old knowledge to shine in a new light (Steinbring, 2005).

Steinbring supplies a “theoretical basis, where the epistemological conditions of mathematical knowledge are particularly related to interactive constructions of knowledge”. (Steinbring, 2005, p. xii). He combines the epistemological triangle as seen in figure 1 with Luhmann’s concept of communication (Steinbring, 2000).

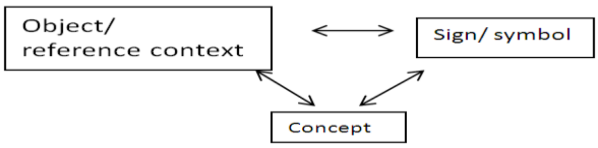

Fig 1: Epistemological triangle (taken from Steinbring (2005, p. 22))

In interaction with others, the students must produce actively reciprocal connections between the “points” of the triangle (Steinbring, 2005). For example while students discuss the concept of functions, they relate the symbol “f” with a diagram as a reference context. But this relation is not fixed; it can be modified during the interaction with others. So the “epistemological triangle reflects the particular status of mathematical knowledge as it has been constructed in the interaction to a certain point of time” (Steinbring, 2005, p. 78)

This view of producing mathematical knowledge through interaction allows us to model “the nature of the (invisible) mathematical knowledge by means of representing the relations and structures constructed by the learner in the interaction" (Steinbring, 2005, p. 23). Moreover, the learning progress of a student or a group in the form of the development of interpretations can be represented as a sequence of epistemological triangles. "In the ongoing development of mathematical knowledge, the interpretations of the sign systems and the appropriately chosen reference contexts are modified or if necessary further generalized by the student" (Steinbring, 2005, p. 23).

THE STUDY

A case study can provide a rich and significant insight into events and behaviours, provide descriptive details about a particular phenomenon, increase understanding of phenomenon and explore uncharted issues (Yin, 2006).

In this paper I will present a case study on students' discussions on one clicker question focussing on learning the mathematical language. The results presented here were part of a larger study in an undergraduate analysis course that almost 100 students attended. In the larger study 16 questions were presented and discussed in four theatre style lecturers each 90 minutes long. Six or seven discussions were recorded per clicker question. The clicker question of this paper (figure 2) was presented at the beginning of the second lesson.

According to Yins' (2014) six differentiations of case studies, this case study was an open participant observation. The students knew that their discussions were recorded and the investigator attended the lecturer but only as a passive bystander.

With the given clicker questions the students had about 6 minutes to discuss the solution with their peers. For analysing the Peer Instruction, the students were asked which group was willing to have their discussion recorded. Seven groups volunteered. But only six of these discussions were useful because one group turned off the dictaphone after 17 seconds.

For the validity and reliability of the case study, as postulated by Yin (2014), the audio recordings of the discussions were transcribed using GAT rules (Breidenstein, 2004). Afterwards the transcripts were interpreted by turn-by-turn analyses among members of the study group (investigator triangulation) as described by Krummheuer(2010). Afterwards, in order to uncover the knowledge construction, it was analysed with Steinbring's epistemology oriented methodology as described above.

The clicker question

One example of the difficulties that undergraduate students face with the understanding typical mathematical

Definition of local maximum

D is a subset for $\mathbb{R}$ and $f: D \to \mathbb{R}$ with $x_o \in D$:

Please choose the definition that fits best to your understanding of a local maximum:

f has a local maximum at x_o if and only if,

A) for all $\varepsilon \in \mathbb{R}^+$ and all $x \in D$ with $|x - x_o| < \varepsilon$ we have: $f(x_0) \geq f(x)$.

B) there exists an $\varepsilon \in \mathbb{R}^+$ such that for all $x \in D$ with $|x - x_o| < \varepsilon$ we have: $f(x_0) \geq f(x)$.

C) for all $\varepsilon \in \mathbb{R}^+$ there exists a $x \in D$ with $|x - x_o| < \varepsilon$ and $f(x_0) \geq f(x)$.

Fig 2: The clicker question

expressions was presented in the paper of Dubinsky and Yiparaki(2000). They presented major problems with the understanding of the interlacing of “for all…there exists” (AE) and “there exist…for all” (EA). “Most students […] could not distinguish between AE and EA statements in mathematics and did not seem to be aware of the standard mathematical conventions for parsing statements” (Dubinsky & Yiparaki, 2000, p. 239). Based on these findings a clicker question was designed and presented in the lectures to teach such expressions for a specific example.

In this question the correct answer B) is contrasted by the two definitions A) and C). In definition A) the students should realize that “all ε ϵ IR” and “all x ϵ D” can be shortened to “all x ϵD” hereby defining an absolute (global) maximum. In definition C) the EA statement were switched around to an AE statement with the result that every point of a function fulfil the requirements of definition C.

ANALYSES OF THE GROUP DISCUSSIONS

In this analysis I will examine three parts of the discussion that the three students Susan, Mike and Lucas had, as an example to show how the clicker question triggered the students’ discussion process and influenced the learning process of the group.

Phase 1: Exchanging the decisions on the first vote

At the beginning of the discussion the three students informed each other about their decision in the first vote and justified it:

8.[2]S:I chose C simply because from our experience in the course, it has always been “for all epsilons”. I don’t know, that was my initial rationale. (1.2)

9. M: hmm (2.0)

10. L: the good old way

11. S: yes but still an explanation.

12. M: are you sure because in principle the idea is (---) that you have the maximum here

13. S: yes

14. M: and principally a kind of curtain that we hang up at the maximum and pull it down (--) and it should be (--) for all epsilon, so that we can create (--) all of them below that distance. That is why I decided on A (3.2)

15. S: hmm (affirmative)

16. L: Well now I am also for definition A. First I decided B to be the correct answer and now I am rejecting this there exist an epsilon because mmm. It doesn’t make any sense if there is only one.

All three students voted for different variants of the clicker question and their approaches differed greatly. Susan voted for C on purely formal reasons. She focused on the expression of “for all εϵIR there exist a xϵD” (AE) in definition C and

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[2] *The numbers of the original transcript have been retained*

concluded that her past experience with definitions and theorems in the lectures, there were only AE expressions.

Mike instead tried to connect his mental picture of a local maximum with the formal definition. His statement influenced Lucas. After he had heard the explanation from Mike he discards definition B and favoured definition A instead.

However Mike's statement had an influence on the group discussion, too. Susan asked for a sketch for a better understanding of Mike's explanation, and the whole group started to compare their conceptual image of a local maximum with the formal definitions of the clicker question.

Phase 2: The comparison of definition A and C

The group has ruled out definition B with the words "there exists one epsilon that must be a joke" very quickly. Afterwards they started to compare definition A) and C) with a sketch in front of them. Mike, who favoured definition A, started with the words:

35. M: here we have our $f(x_0)$ and (--) x minus x_0 is smaller than epsilon must define the interval where for all x the f(x) must be smaller because if…..
36. S: But wouldn't it be all x
37. M: no no that's true, wait. You're right (---)
38. S: if it's valid for all x then it would be really big, don't you think?
39. M: no no it says it's valid for all x within this interval
40. L: Yes that is definition A (---) because if it is valid for only one x like in definition C that doesn't work.
41. M: because otherwise there could be something higher next to it.

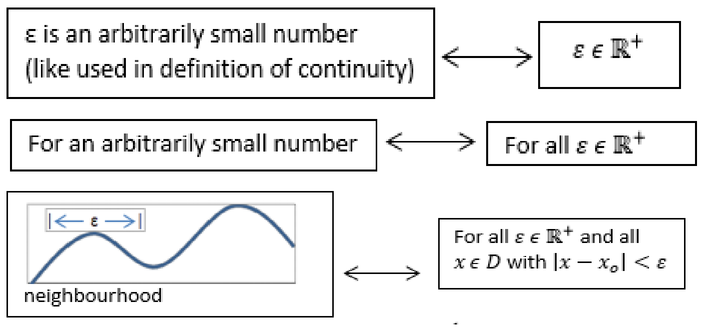

Fig 3: Mikes three misinterpretations

The group mainly focussed on the two statements in definitions A and C: "there exists a $x\epsilon D$" (Def. C) and "for all $x\epsilon D$" (Def. A). The students tried to understand the impact of the differences on the meaning of the definitions.

In turn 35 Mike started with his interpretation of definition A by creating a sketch in front of them. His assumption that definition A is correct was based on three misinterpretations he expressed before. From his previous experience in the analysis course, he connected the sign/symbol "$\varepsilon \in IR^+$" in definition A with the idea of an arbitrarily small number ε. The other misinterpretations were the result out of the first one. The mathematical symbol "for all"$\varepsilon \in IR^+$" was connected to the idea (reference context) of “for arbitrary small numbers” which in return resulted in the interpretation of the sign/symbol “for all$\varepsilon \in$ IR^{+}” and all $x \in D$ with $|x - x_0| < \varepsilon$” as an ε-neighbourhood of x_0 (illustrated in fig. 3).

Susan’s objections in turn 36 and 38 was based on the focus on the expression of “for all x ϵ D” in definition A. At this moment she didn’t see the connection between the two expressions “for all$\varepsilon \in$ IR^{+}” and “for all x ϵ D.” However, her words helped Mike to focus on the connection between these two statements. When he tried to argue against Susan’s objections he started to recognise his mistakes illustrated in figure 3:

54. M: But the problem is….what if here we, if here we ummm(--)
55. S: That’s why you have your epsilons here, right?
56. L: no that is for every epsilon
57. M: but if it’s valid for every epsilon then we aren’t getting any closer here.

Mike’s words in line 54 ends with an eureka moment. Suddenly he realised his misinterpretation and in line 57 he tells his fellow students his new view of definition A. This new view ends up with the realisation that definition A defines a global maximum

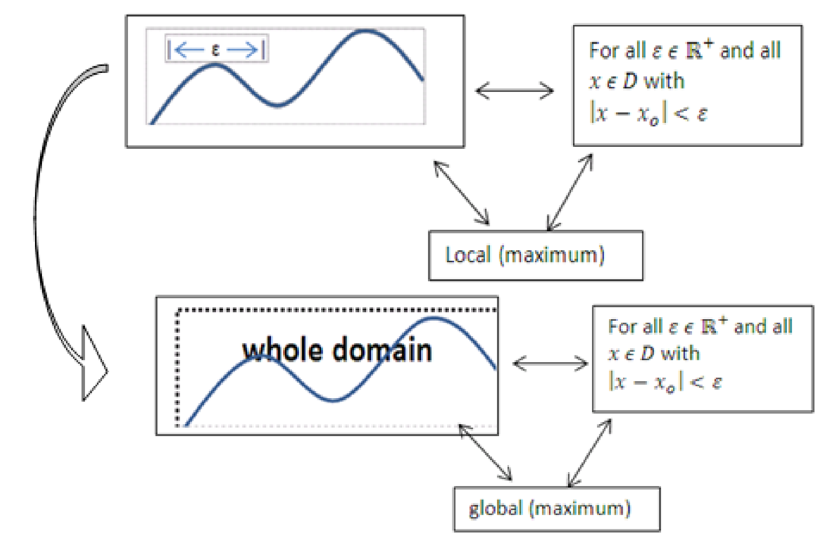

Figure 4: The shift of interpretation of definition A

M: if it is for all epsilon and you choose this as the maximum, then you don’t get this one (--) because you say it’s for all epsilons. But for a global maximum it obviously works.

This shift is illustrated in figure 4 with the epistemological triangle. Now the sign “for all$\varepsilon \in$ IR^{+}“ and all $x \in D$” with $|x - x_o| < \varepsilon$” was connected with the concept of“global” and the reference context illustrated in figure 4.

Phase 3: Understanding definition B

After the group recognised definition A as a definition for a global maximum Mike and Susan interpreted definition C:

77. M: […] I am almost convinced to say definition C is correct because of the expression there exist one x (--) I think you can find always an x for every small neighbourhood. No matter how close you get to x0, you always find an x that is smaller.
78. S: Yes [so far as you say

79. M: [and that is for a local maximums

80. S: yes that makes sense, because the maximum is the highest (--) and at least there must exist a x that is smaller.

81. M: exactly because the maximum is local (--) I will try the definition with one x now. I would say C now.

82. S: OK good then I'll stick with C too

83. L: I think I will go with choice B

84. S: what is your idea behind B

85. L: I would explain it this way if at this point you can find any interval that was B

86. S: yes

87. L: You find any interval so that they are all smaller, then you have a local maximum and that is exactly what is stated in B: Find an epsilon interval around this and they must all be smaller. That is exactly how it is formulated in B.

88. S: Then you would have a solid epsilon.

89. L. Yes, you only have to find one.

90. M: YES, you're right.

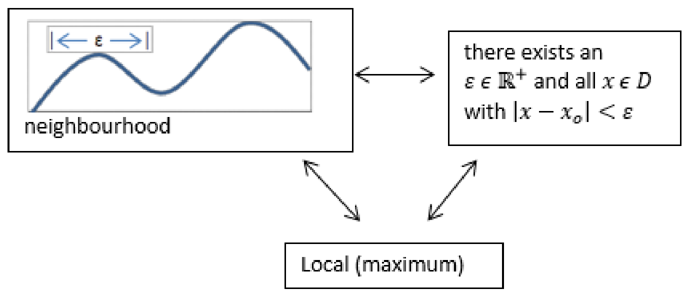

Figure 5: New interpretation of "local"

When Mike and Susan talked about definition C, they used their imagination and the sketch of a local maximum in front of them. They tried to figure out if a local maximum fulfils the requirements that are stated in definition C. The result is that the maximum of their sketch meets the necessary requirements of definition C. Thus they decide for C although Sarah does not seem to be completely convinced (line 82).

Then in line 83 Lucas surprisingly proclaimed definition B to be correct. Before he was quiet and didn't argue with the other about definition C. He had used the small break to think individually about the question. He explained his decision for B in line 87. Susan was surprised and Max agreed with the words "Yes, you are right". Eventually Lucas explanation leads to a new interpretation of definition B and the concept of local (see figure 5).

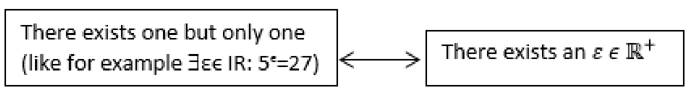

Figure 6: Interpretation of the sign "there exists anε ϵ IR$^+$"at the beginning of the discussion

The group had ruled out definition B to be correct because of their misinterpretation of the sign "there exists anε ϵ IR$^+$" (see figure 6) andfinally found the correct definition for a local maximum at the end of the discussion.

CONCLUSIONS AND DISCUSSION

The clicker question was designed to help the students to understand the different meanings of the expressions of AE and EA statements in a mathematical context as recommended by Dybinsky and Yiparaki(2000). As seen in the discussion many meaning making situations were triggered by different interpretations of the variants of the multiple choice question (mc question).

The change of the interlacing of “for all…” and “there exist…” statements between the variants of the mc questions had a great influence on the meaning of the three definitions. The students had to work out these differences. During that process misunderstandings and misinterpretations were seen. According to Muller (2008), the addressing of misunderstandings is an important part to overcome such misunderstandings. Definition C started with the AE statement like many other definitions[3] in the course before. So definition C was able to unveil Susan’s generalisation that any definition with quantifiers had to start like this.

The “for all ε ϵ IR” statement in definition A was interpreted by Mike as an arbitrarily small number. One explanation for such an interpretation is the common use of ε in the course before, like in the definition of convergence of sequences. This definition starts with the statement “for all ε ϵ IR” but it is just “used” in the way expressed by one student in another group during their discussion:

> **N**: definition A makes most sense for me, it means you approach over all x but let the interval get smaller and smaller. I see a connection to the concept of convergence (.) that you shorten the distance more and more (1.0) nevertheless the $f(x_0)$ is the greatest.

Despite these difficulties, the students were able to find the right answer at the end of the discussion. The key for the construction of new mathematical knowledge was the attempt to find connections between the mathematical symbolic expressions and their conceptual image, as well as to discuss different interpretations. Phase 2 is a particularly good example of this. Both Susan and Mike were making mistakes, but together they influenced each other in a positive way. Susan’s objection and Mike’s counterarguments led to a new view and understanding of the meaning of definition A (figure 4).

According to Goos et al. (1996) three factors influence the quality of collaborative mathematical discussions in school (see figure 6): the task has to be for learning and not for performance, the students should have equal task specific expertise and the task should be challenging for all students (Goos et al., 1996, p. 243). These conditions were met here. None of the students in the group knew the right answer at the beginning of the discussion and solving the task was challenging for all of them. Therefore, this leads to the hypothesis that these factors are influencing the quality of PI at University as well.

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[3] *Like the definition of convergence of sequences and the definition of continuity*

Moreover the analysis of the discussion shows that the presented clicker question complies with the four demands for tasks to support collaborative learning in some way: meaningful, complex, need for different ability to be solved and aim of level raising (Dekker & Elshout-Mohr, 1998).

This question was complex and difficult enough to encourage the involved students to debate the meaning of the different expressions (complex). But it was not too difficult in comparison to the students' skills and knowledge. The students used their conceptual image of a local maximum that they learnt during high school in order to work out the different meanings. Their different interpretations of the AE and EA expressions helped the group to find the right answer (different ability) and the construction of new mathematical knowledge (level raising).

However, one has to considerwhether the quality of the discussion depends on the interplay between the clicker question and the skills, motivations and knowledge of students in the group. The impact on such questions on the quality of the discussion in relation to different group dynamics should be investigated further because lecturers have to find questions that challenge as many students as possible. The analysis of all six recorded discussions is an encouraging sign that these kind of questions can trigger high quality discussions in different kind group compositions as well: one discussion failed totally because one student were afraid to say something wrong but four of the remaining five groups were also able to construe new mathematical knowledge during the discussion.

Clicker questions like the one presented here could be implemented more often during undergraduate analysis courses. For example a question could be designed on the definition of convergence of sequences or continuity in the same way. Then such clicker questions might be one pedagogical tool to learn the mathematical language as desired by Nardi(2011, p. 2056).

REFERENCES

Breidenstein, G. (2004). Peer-Interaktion und Peer-Kultur. In W. B. Helsper, Jeanette (Ed.), Handbuch der Schulforschung (pp. 945-964).

Dekker, R., & Elshout-Mohr, M. (1998). A Process Model for Interaction and Mathematical Level Raising. Educational Studies in mathematics, 35, 303-314.

Deslauriers, L., Schelew, E., & Wieman, C. (2011). Improved Learning in a Large-Enrollment Physics Class. Science, 332, 862-864.

Dubinsky, E., & Yiparaki, O. (2000). On Student Understanding of AE and EA Quantification. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), CMBS Conference Board of the Mathematical Sciences (Vol. 8, pp. 239-289). Wahington, D.C.

Freeman, S., Eddy, S. L., McDonougha, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. Proceedings of the National Academy of Sciences, 111(23), 8410–8415. doi:10.1073/pnas.1319030111

Goos, M., Galbright, P., & Renshaw, P. (1996). When Does Student Talk Become Collaborative Mathematical Discussion? In P. Clarkson (Ed.), Proceedings of the 19th annual conference of the Mathematics Education Research Group of Australasia Gold Case (pp. 237-244). Melbourne: MERGA.

Hake, R. R. (1998). Interactive-engagement versus traditional methods: A six-thousand-student survey of mechanics test data for introductory physics courses. American Journal of Physics Teachers, 66(1), 64-74.

Knight, J. K., Wise, S. B., & Southard, K. M. (2013). Understanding Clicker Discussions: Student Reasoning and the Impact of Instructional Cues. CBE-Life Sciences Education, 12(4), 645-654. doi:10.1187/cbe.13-05-0090

Krummheuer, G. (2010). Die Interaktionsanalyse. Retrieved from http://www.fallarchiv.uni-kassel.de/wp-content/uploads/2010/07/krummheuer_inhaltsanalyse.pdf

Mazur, E. (1997). Peer Instruction: A User's Manual. Upper Saddle River, NJ: Prentice Hall.

Muller, D. A. (2008). Designing Effective Multimedia for Physics Education. (Doctor of Philosophy), University of Sydney.

Nardi, E. (2011). 'Driving noticing' yet 'risking precision': University mathematicians' pedagogical perspectives on verbalisation in mathematics. Paper presented at the Proceedings of the 7th Conference of European Researchers in Mathematics Education, Rzeszow, Poland.

Schichl, H., & Steinbauer, R. (2009). Einführung in das mathematische Arbeiten. Heidelberg, Berlin: Springer.

Smith, M. K., Wood, W. B., Adams, W. K., Wieman, C., Knight, J. K., & Su, T. T. (2009). Why Peer Discussion Improves Student Performance on In-Class Concept Questions. Science, 323, 122-124. doi:10.1126/science.1165919

Smith, M. K., Wood, W. B., Krauter, K., & Knight, J. K. (2011). Combining Peer Discussion with Instructor Explanation Increases Student Learning from In-Class Concept Questions. CBE-Life Sciences Education, 10(1), 55-63.

Steinbring, H. (2000). Mathematische Bedeutung als eine soziale Konstruktion – Grundzüge der epistemologisch orientierten mathematischen Interaktionsforschung. Journal für Mathematik–Didaktik, 21(1), 28-49.

Steinbring, H. (2005). The Construction of New Mathematical Knowledge in Classroom Interaction (Mathematics Education Library ed. Vol. 38). Berlin, New York: Springer.

Yin, R. K. (2006). Case Study Model. In J. L. Green, G. Camilli, P. B. Elmore, A. Skukauskeite, & E. Grace (Eds.), Handbook of complementary methods in education research. Washington D.C.: American Educational Research Association.

Yin, R. K. (2014). Case study research. Design and methods. Thousands Oaks: Sage Publications.