# Concept Images of Open Sets in Metric Spaces

Safia Hamza and Ann O'Shea

Department of Mathematics and Statistics, Maynooth University, Ireland, ann.oshea@nuim.ie.

*We consider the concept images of open sets in a metric space setting held by some Pure Mathematics students in the penultimate year of their undergraduate degree. Ten students were interviewed and asked to define the concept of an open set, as well as to work on some specially designed mathematical tasks on this topic. The analysis of the interview data revealed five main categories of concept image of open sets based on: the formal definition; the idea of boundary of sets; open sets in Euclidean space; the union of open balls; visualisation.*

*Keywords: concept image, concept definition, open sets, topology.*

## INTRODUCTION

This paper concerns a study of the conceptions held by students taking a module on metric space topology. The main topic of interest is the notion of an open set in a metric space. This concept is fundamental in the study of topology but (personal) experience has shown that it can pose problems for students and hinder the development of their understanding of the subject. Our goal was to explore the students' concept definitions and concept images of the notion of open set in order to provide information to lecturers which would help them when planning and delivering courses in this area.

Courses on Metric Spaces often involve significant transitions for students. These students usually have taken a course in analysis on the real line but may not be comfortable with the level of abstraction required to work in general metric spaces. Part of this transition involves coming to an appreciation of the role of definitions in abstract mathematics; Edwards and Ward (2004) investigated students' understanding and use of definitions in an introductory abstract algebra course and found that students seem to place less emphasis on definitions than mathematicians would, and even when they are able to correctly state a definition of a concept they may not always use this when working on problems. Very little research has been carried out on students understanding of topics in introductory topology and we wanted to gain information about how students define concepts in this area and also to explore the concept images related to these notions. Our research question was:

- What elements of students' concept image of open sets in metric spaces can we identify?

## THEORETICAL FRAMEWORK

We will use Tall and Vinner's (1981) description of the notions of concept definition and concept image. They used the term *concept definition* to indicate a mathematical definition:

> a form of words used to specify that concept (Tall and Vinner 1981, p. 152).

They used the term *concept image* to mean all that an individual has in his/her mind about a concept, and this would include mental pictures, experiences and impressions that are associated with it. They defined the concept image as:

> the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. (Tall and Vinner 1981, p. 152).

They explained also that a concept image is not a static item in memory; it builds and is reconstructed over time as individuals meet new stimuli. Tall and Vinner (1981, p. 152) also used the term *evoked concept image* to describe the part of a concept image which is evoked by the concept name at a specific time.

Since the introduction of these notions, they have been used in many research studies (see Alcock and Simpson (2009) for an overview) to understand the development of understanding of various concepts in for example calculus (Bingolbali and Monaghan, 2008), linear algebra (Wawro, Sweeney, & Rabin, 2011), and introductory real analysis (Przenioslo, 2004). To the best of our knowledge there has not yet been a study of the concept images of concepts in general topology.

Przenioslo (2004) studied students' concept images of limits of functions and amongst other results she found that aspects of concept images can be formed very early in a student's development. McGowen and Tall (2010) also addressed the role of early experience (met-before) on the learning of mathematics. They described that

> The term met-before applies to all current knowledge that arises through previous experience, both positive and negative. It can be given a working definition as 'a mental structure that we have *now* as a result of experiences we have met-before'. (McGowen & Tall 2010, p. 171).

They explained that previous experience could be supportive (in which the old ideas can make sense in the new context) but could also be problematic.

Fischbein (1989) also referred to the positive and negative effects of previous experience on mathematical reasoning and understanding. He spoke about 'tacit models' as models of abstract concepts which are developed early in the learning process and which continue to influence reasoning and interpretation without the learner being explicitly aware of this influence. Problems occur when a tacit model, or possibly a specific example, becomes a substitute in the learner's mind for the concept in question. If the learner is not aware of the influence of these models and examples on their own thinking, then they can do little to change them. Fischbein (1989) suggests that researchers should therefore investigate the likely tacit models related to a concept, and that teachers should make students aware of the existence of these models and of the problems they may cause; the aim of both should be to provide students with opportunities to recognise and control their own tacit models.

Bingolbali and Monaghan (2008) observed that many of the learning theories that developed using the construct of Tall and Vinner (1981) of the concept image and concept definition were cognitive theories of learning. They argued that the construct could be also used to study social theories of learning. They studied first year Mechanical Engineering and Mathematics students' concept images of the derivative, in particular the rate of change and tangent notions. They showed that students' development of their concept image is affected by the teaching practices and by their departmental affiliations.

## METHODOLOGY

This study involved students taking a course on Metric Spaces at Maynooth University in Ireland. This was a one semester module which was delivered by an experienced member of staff. The authors had access to the course notes and assignments. The course ran for twelve weeks and there were two lectures each week. The syllabus for this course is: Metric spaces: definitions and examples, convergence and continuity in metric spaces; uniform continuity; pointwise and uniform convergence, open and closed sets; basic properties; continuity in terms of open sets; limit points; closure; interior and boundary, completeness and compactness.

All 17 students in the module were asked to participate in this study and 10 volunteered to be interviewed. The interviews took place in the final two weeks of the semester. The interviews were conducted by the first author and were task-based (Goldin, 1997) and semi-structured. They were audio-recorded and fully transcribed; the data was anonymised immediately. We will refer to the 10 students using the letters Q - Z. After some initial introductory questions, the students were asked to define an open set in a metric space and how they would explain the concept to a friend. They were also asked to work on some tasks. The tasks used were designed for the study taking care to use the same language and notation as that employed by the module lecturer; they were piloted in written form by two recent graduates. Four tasks were designed for this part of the study but we will only report on two of them here. These tasks were:

A. Consider the metric space $(\boldsymbol{Z}, d_{\boldsymbol{Z}})$, where $d_{\boldsymbol{Z}}$ is the standard metric inherited from $\boldsymbol{R}$, and let $B = \{m-1, m, m+1\}$. Is $B$ an open ball in $(\boldsymbol{Z}, d_{\boldsymbol{Z}})$? If your answer is yes, please specify the centre and radius of the ball. If your answer is no, please explain. Can you find an open ball $C$ which is a subset of $B$?

B. Let X be the set of all real sequences. Define:

$$d(\{a_k\},\{b_k\}) = \begin{cases} 0 & if \quad a_k = b_k \ for\ all\ k \in N \\ 1/k & if \quad k = \min\{n \in N : a_n \neq b_n\} \end{cases}$$

(i) Can you describe this metric in words?
(ii) What do you think this metric measures?

(iii) Let $\{0\} = \{0,0,0,\ldots\}$. If $d(\{a_n\},\{0\}) = 1$, what can you say about $\{a_n\}$?

(iv) What is $B(\{0\},1)$? What is $B(\{0\},{}^{1}/_{2})$?

(v) Is the set of sequences $\{\{a_n\}: a_1 = 0\}$ open? Is the set of sequences $\{\{a_n\}: a_1 = 0 \text{ } or \text{ } 1\}$ open?

(Note that the lecturer had defined $B(a,r) = \{x \in X | \, d(a,x) < r\}$ in the metric space *(X,d).*) The transcripts were analysed using a grounded theory approach (Strauss and Corbin, 1990) by both authors independently, the codes and categories created were then compared and a final coding was agreed.

## RESULTS

### Students' Definitions

The students were asked:

(i) To define the term *open set* in a metric space,

(ii) How they would explain this term to a friend.

We analysed the answers to these questions and classified them into three categories. These were: answers based on the formal definition of an open set; answers based on the notion of an open set as a union of open balls; answers related to the boundary of a set. The formal definition given by the lecturer in this course was:

*A subset U of a metric space (X,d) is an open set if for all $x \in U$ there exists ε(x)>0 and an open ball B(x,ε(x)) in (X,d) such that B(x,ε(x)) is a subset of U.*

None of the students gave exactly this definition but some gave something very close to it. For example Student Q said

> The set is open if for any point in the set you can draw an open ball around it which is contained in the set.

Students X and Z gave similar definitions. Student Y said

> The official definition is you can take any open ball around any point and it's still completely contained in the set.

We can see that this is not correct as it is too strong; we do not need every open ball centered at every point to be a subset of U, we just need at least one for every point. Notice that none of the students spoke about the ball being open in (X,d), we assume that they mean this implicitly.

When asked to define an open set, Students S, T, U, V and Z all said that open sets were unions of open balls. Note that Student T was the only student who seemed to realise that this is a theorem and not a definition and she used the formal definition in her explanation to a friend.

The last category of definition is made up of answers to Question (i) which mention boundaries when trying to define the term open set. Student R said

> Open set – something which doesn't have a clear boundary, you can get as close as you like but never get to the actual end of the set.

Student W gave a similar definition after first admitting that he had forgotten the formal definition. He said

> We can say what the general idea, the open set is basically, it isn't like say straight edges, is kinda fuzz out, because it doesn't contain border elements

and in his explanation to a friend he also said

> so it kind of fades off infinitesimally close to boundary, but it never quite gets out, fuzzy at the edges

| ***Student*** | ***Answer to Question (i)*** | ***Answer to Question (ii)*** |
|---|---|---|
| Q | Formal Definition | Formal Definition |
| R | Boundary | Boundary |
| S | Union of Open Balls | Union of Open Balls |
| T | Union of Open Balls | Formal Definition |
| U | Union of Open Balls | Boundary |
| V | Union of Open Balls | Union of Open Balls |
| W | Boundary | Boundary |
| X | Formal Definition | Formal Definition and Union |
| Y | Formal Definition | Boundary |
| Z | Union and Formal Definition | Boundary |

**Table 1: Students answers to the definition questions**

Table 1 shows the category of answer given by each of the students to both questions. It shows that in answering question (i) three students gave a definition close to the formal definition, four spoke about unions of open balls, one student mentioned both these ideas, but only two students mentioned anything to do with boundaries. This is in contrast to the answers given in part (ii) where five students spoke about boundaries, two gave answers based on the formal definition and two explained using unions of open balls, with one student using both the formal definition and the idea of unions. It may be that when seeking an explanation suitable for a friend, students looked for examples or non-mathematical terms to illustrate the idea and that this led them to concentrate on boundaries or the lack of them. In answer to Question (ii) Student Y said

> We got a definition for it last year which is just a set that doesn't contain its boundary. So it's kinda easiest to think of it in that way, I probably explain it kinda like that, that you know, if you go shorter and shorter distance so you know, no matter how close you get, you'll never quite get there.

These students had taken an introductory course on analysis on the real line and it may be that their experience with open intervals there has influenced their definition of openness in a metric space; that is they may have a tacit model of the concept of open sets based on examples familiar from the earlier course.

## Students' Concept Images

The answers to Questions (i) and (ii) above show us that the students' concept image of an open set in a metric space includes more than the formal definition and in particular includes results proved about open sets (i.e. that open sets can be expressed as unions of open balls) and previous experience of sets without boundary. We analysed the students' answers to the mathematical tasks in the interviews in order to see if other aspects of the evoked concept image of open sets would emerge. We found that some students used the formal definition when working on these problems and also frequently referred to boundaries but no one used the notion of unions of open balls (which would have been very useful in Task B). The other aspects of concept image that we observed included visualisation of open sets using generic pictures (like discs) and notions related to open sets (particularly open intervals) on the real line. We will give some examples of these two aspects of evoked concept image here (to save space we will not revisit the other aspects of concept image encountered in the last section).

Many of the students spoke about visualising the open sets in the tasks. For most students their picture of an open ball seems to be based on pictures from Euclidean space. For example Student Q when considering Task A commented that:

> *Student Q* B is open if we can draw an open ball around 1 which is inside the set, if its centred at 1 then the open ball would include 0 and 2.
>
> *Interviewer* you said here, draw an open ball, so that means you have a picture of open ball in your mind?
>
> *Student Q* yeah!
>
> *Interviewer* which kind?
>
> *Student Q* just a ball, a circle, um it has to be inside the set.

Similarly many of the other students drew circular regions when thinking about open sets, for example Students Q and S did this on Task B (see Figure 1). Indeed, some students expressed frustration with that problem because they had difficulty in visualising the open sets concerned (Students R, T, and Y).

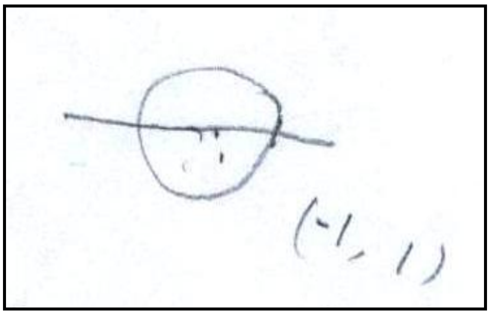


**Figure 1: Student S's picture of B({0}, 1) in Task B**

Some of the students also spoke about having difficulty with other questions when they couldn't visualise the sets in question. For example Student X, when considering Task A said:

> I don't think I have seen a set like that and been asked if it is an open ball, so I can't really picture it.

Some students seemed to realise that their intuitive pictures may not help them in all situations. For example Student Y initially tried to use diagrams to answer Part (iv) of Task B, however later she returned to the definition of the metric $d$ and worked with that analytically. When asked why she did that she answered:

> Because I just looked at it to, and it looked too confusing to try and think of a picture of sequences, to try to think of how far they're apart.

It seems that students' concept image of open sets contains visual elements and some of these are based on open sets in familiar metric spaces especially Euclidean space. This space, and the features of open sets in it, appears to have other influences on the students' concept images of open sets. In Task A, some students were reluctant to see $B$ as $B(m,1+\varepsilon)$ because it consisted of (apparently) isolated points. Student T said

> I think I'm going that, r is 1 and x is centre m. But, no, that is not open because it doesn't contain all the points. It's only contains these three points, it's limited, meets these three points, I don't think it's open.

Recall that Student W referred to 'fuzziness' when defining an open set. He used this idea here again

> We'd only got three elements, but these elements all have space of the exactly one. So you either have a gap of 0 or 1 between them. There is no kind of fuzziness in between, so you can't make it open. Like, it'll either contain them or not.

These students may be referring to properties of open intervals in $\boldsymbol{R}$ or open discs in $\boldsymbol{R}^2$ such as connectedness and completeness. We saw this idea in the students' answers to other tasks too.

## SUMMARY AND DISCUSSION

We found that the students in this study had three main ways of defining open sets in metric spaces: the formal definition; using the notion of boundary; and using the fact that open sets can be expressed as unions of open balls. Our analysis has also showed that these students had varied concept images related to the open set concept. These concept images were based on: the formal definition, the boundary idea, unions of open balls, openness in Euclidean space, and visualisation. When working on mathematical tasks students used both the definition and other aspects of their concept image. We noted that the students who routinely based their reasoning on the definition were more successful when working on problems; this was especially true in Problem B where students' unfamiliarity with the context meant that some components of their concept image did not help them. Most students showed a richness in their concept image and an ability to view open sets in a variety of ways.

We noticed that the students' confusion about boundary points and endpoints of an open set could cause difficulties. Moreover we noticed that the previous experience of open sets in $\mathbf{R}^n$ has an effect on students' understanding of openness in general metric spaces. We also observed that some students used their visualisation of an open ball as a circle or a disc and they appeared to base their reasoning on this when thinking of open balls. This echoes the findings of Przenioslo (2004) and McGowen and Tall (2010) that students' previous experience can influence their thinking in significant ways. It seems that some of the students may have tacit models (Fischbein 1989) of open sets based on their previous experience which influences their reasoning without their explicit knowledge. This influence can be very positive and can help students build intuition and develop understanding however it may also cause difficulties. It is important for lecturers to realise this point when introducing new concepts. Indeed McGowen and Tall (2010) suggested that mathematicians should not only consider the positive influence of students' prior learning on their understanding of a new concept but also should address the possible ways in which it could hinder the learning process. For example lecturers could be careful to introduce students to a variety of examples of metric spaces and to point out the differences between them and the more familiar Euclidean space. From our analysis of the course materials, it seems that the lecturer worked hard on this by using examples from a wide class of metric spaces, but we see that the effects of previous experience still persist.

Wawro et al. (2011) reported on the ways in which students' definition of subspaces in a Linear Algebra course were integrated into their concept image of the concept. They found that students often had both geometric and algebraic aspects in their concept image and they saw that encouraging students to work with the definition was successful in overcoming potential cognitive conflicts or inconsistencies. One possible way forward is to employ the Defining as a Mathematical Activity framework of Zandieh and Rassmusen (2010) which aims to provide a means of creating rich links between concept images and concept definitions. An approach like this may be fruitful in helping students explore the meaning of, for example, a boundary in metric spaces.

This study has given us some information about the definitions that the students in this study use and the components of their concept images in metric space topology. We believe that this information would be useful to lecturers when planning and delivering courses in this area. We make no claim that our results are generalizable to all topology students; however in the spirit of Fischbein (1989) we hope that the findings could be at least used to alert learners to tacit models or aspects of their concept images that may be limiting their understanding and reasoning. The study presented here was relatively small in scale and it would be interesting if it could be extended to students in other universities to see if additional conceptions of openness appear.

We have further data about students' concept definitions and concept images of the notion of distance in a metric space and we hope to report on this soon.

## REFERENCES


Alcock, L. and Simpson, A. (2009). *Ideas from mathematics education, an Introduction for Mathematics*, The Higher Education Academy: MSOR network, Birmingham.

Bingolbali, E. and Monaghan, J. (2008). Concept image revisited, *Educational Studies in Mathematics*, 68 (1), 19-35.

Edward, B. S. and Ward, M. B. (2004). Surprises from mathematics education research: Student (mis)use of mathematical definitions, *The American Mathematical Monthly*, 111 (5), 411- 424.

Fischbein, E. (1989). Tacit Models and Mathematical Reasoning, *For the Learning of Mathematics*, 9(2), 9-14.

Goldin, G. A. (1997). Observing mathematical problem solving through task-based interviews, *Journal for Research in Mathematics Education. Monograph: Qualitative Research Methods in Mathematics Education*, 9, 40-62 + 164-177.

McGowen, M., A. and Tall, D. O. (2010). Metaphor or met-before? The effects of previous experience on practice and theory of learning mathematics, *Journal of Mathematical Behavior*, 29 (3), 169-179.

Przenioslo, M. (2004). Images of the limit of function formed in the course of mathematical studies at the university, *Educational Studies in Mathematics*, 55, 103-132.

Strauss, A. and Corbin, J. (1990). *Basics of qualitative research*: Grounded theory procedures and techniques, *Sage*, London.

Tall, D. And Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12 (2), 151-169.

Wawro, M., Sweeney, G., F. and Rabin, J., M. (2011). Subspace in linear algebra: Investigating students' concept images and interactions with the formal definition, *Educational Studies in Mathematics*, 78 (1), 1-19.

Zandieh, M. and Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning, *The Journal of Mathematical Behavior*, 29, 57-75.