Didactical implications of using various methods to evaluate $\zeta(2)$

Margo Kondratieva

Memorial University, mkondra@mun.ca

Keywords: multiple proofs, interconnecting problems, Basel problem.

Mathematics instruction may benefit from using *interconnecting problems*, defined as problems that: allow various solutions at both (relatively) elementary and more advanced levels; can be solved by various mathematical tools from different mathematical branches; and, can be used in different courses (Kondratieva, 2011).

This research project uses the following interconnecting proof-problem: Prove that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$. Five solutions are considered, via (1) Euler's representation for $\frac{\sin x}{x}$; (2) integral calculus and trigonometry; (3) reduction to a double integral; (4) Fourier series; and, (5) Cauchy's Residue Theorem. Solution (1) has a historical value, (2) and (3) are suitable for calculus courses, while solutions (4) and (5) require more advanced techniques from analysis. Known as the Basel problem, or evaluation of $\zeta(2)$, this problem has played a stimulating role in mathematical research from the 17th century to this day. It might play a similar role in the teaching of mathematics.

Multiplicity of approaches is important in mathematics research, and therefore, in training future researchers. According to Sir Michael Atiyah, "any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalize in different directions - they are not just repetitions of each other." (Interview in EMS Newsletter, Sept. 2004, p. 24; http://www.ems-ph.org/journals/newsletter/pdf/2004-09-53.pdf) From the educational perspective, studying multiple solutions contributes to a learner's cognitive development through formation of related crystalline concepts (Tall, Yevdokimov, Koichu, Whiteley, Kondratieva, Cheng, 2012, p. 20). In particular, within the "Structure of Observed Learning Outcomes" (SOLO) model proposed by Biggs & Collis (1982), the fundamental UMR-cycle of concepts' construction includes three levels: (U) uni-structural, (M) multi-structural and (R) relational. The SOLO taxonomy focuses attention upon structure of learners' responses. At the U-level, the learner shows familiarity with only one solution. At the M-level, several approaches are used by the learner, without any relation perceived between them. At the R-level, the learner is able to compare, relate and integrate different approaches and ideas. Thus, the use of interconnecting problems may bring the learner to the M-level and help to create the environment in which learners can develop connections while moving towards the R-level. Local UMR-cycles occur in all SOLO-modes of operations, ranging from sensori-motor and iconic to concrete symbolic, formal and post-formal, thus making this framework applicable for analysis at the post-secondary level. A desirable outcome of a UMR-cycle is an extended abstract response, which comprises possible generalizations and extensions.

It emerges from the relational understanding achieved by the learner at the R-level, and often signifies a learner's transition to the higher SOLO-mode of operation. Thus, familiarity with as many as possible approaches, each of which highlights different aspects, properties and contexts where a (interconnecting) problem is considered, may prove vital for learners' cognitive growth.

The interconnecting problem used in this study provides the richness of mathematical context suitable for the emergence of aforementioned UMR-cycle. This preliminary study examines the following questions: (I) To what extent students who have completed a Bachelor degree with a major in mathematics are familiar with each of the five solutions? (II) What are mathematics instructors' views on the possibility to include these solutions into existing university courses? Eight graduate students and 8 instructors of mathematics (all from Memorial University), who have (respectively) completed or taught calculus and analysis courses responded to a survey. The participants were asked to read each of the five solutions and identify whether the solution is (a) familiar (b) accessible (c) connected to other solutions. The participation in the project was voluntary and anonymous. All students showed either U-level or M-level response with at most 3 different solutions recognized as being familiar. While all five solutions were found accessible by all students, advanced solutions (4) and (5) were more popular than solutions (1) - (3). This situation is in agreement with instructors' responses. They favoured solutions (4) and (5) for inclusion in corresponding analysis courses as opposed to presence of solutions (1) - (3) in any courses. The instructors felt that (4) or (5) is a more natural fit, while the other proofs would require extra time for explaining technical details in a course with an already busy curriculum. This challenge outweighed the advantage of giving a broader and more complete picture based on historical material and illustration of alternative techniques, let alone making connections between various methods.

The goal of my paper is to attract readers' attention to this rather not uncommon situation within undergraduate mathematics teaching and hopefully initiate some shift towards a more balanced approach based on inclusion of interconnecting problems in many levels of the mathematics curriculum and linkage of their various solutions.

REFERENCES

Biggs, J. & Collis, K. (1982). Evaluating the Quality of Learning: the SOLO Taxonomy. New York: Academic Press.

Kondratieva, M. (2011) The promise of interconnecting problems for enriching students' experiences in mathematics. *Montana Mathematics Enthusiast*, 8 (1-2), 355-382.

Tall, D., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M. & Cheng, Y.H. (2012). Cognitive Development of Proof. In G. Hanna & M. De Villiers, (Eds.), *Proof and Proving in Mathematics Education: The 19th ICMI Study* (pp. 13-49). New York: Springer.