Making sense of students' sense making through the lens of the structural abstraction framework

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In this paper we use the evolving framework of structural abstraction as a theoretical lens to investigate how mathematics major university students understand the limit concept of a sequence. To this aim the theoretical framework is outlined and previous empirical data on one individual's partial (re-)construction of a convergent sequence is revisited. In doing so, we provide insights in how students, who consider the formal definition of a mathematical concept as one of the components of their concept image, involve it into their overall mathematical discourse when building new knowledge. Deeper analysis also reveals unsettled issues about structural abstraction and provides new directions for advancing our understanding of this kind of abstraction.

Keywords: generic representation, mathematical learning, sense making, structural abstraction, theory development.

INTRODUCTION

There has been a growing interest in revisiting the notion of abstraction in mathematics education. Recent contributions from socio-cultural perspectives on the learning of mathematics have strengthened our theoretical understanding and framed our empirical investigation on abstraction in knowing and learning mathematics, as Hershkowitz, Schwarz, and Dreyfus' (2001) *abstraction in context* approach and Noss and Hoyles' (1996) *situated abstraction* approach indicated. With regard to cognitive approaches on abstraction in mathematics education, Scheiner (2016) observed that the literature demonstrated substantial progress in explicating the significance of Piaget's (1977/2001) *reflective abstraction* in mathematical concept construction, the kind of abstraction that is often described in terms of forming a (structural) concept from an (operational) process (see Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991). However, in the past, the literature rarely explored differences in cognitive processes with regard to whether the primary focus is on the actions (abstraction from actions) or on the objects (abstraction from objects). The former takes place on the actions on objects, in particular, individual's reflections on actions on known objects; the latter takes place on the objects themselves, in particular, paying attention to the properties and structures inherent in those objects. However, Piaget considered abstraction from actions as the only form of abstraction for mathematical epistemology; separating it from abstraction from objects. Given these historical origins of our field, it is not surprising that the literature reveals a bias towards abstraction from actions as the dominating form of abstraction in knowing and learning mathematics.

Only recently, abstraction from objects has attracted attention as a form of abstraction that provides an account for the complex cognitive processes compatible with students' sense-making strategy of 'giving meaning' (Scheiner, 2016; Scheiner & Pinto, 2014). An important contribution within abstraction from objects has been provided by Mitchelmore and White (2007) who investigated *empirical abstraction* in learning elementary mathematics drawing on Skemp's (1986) conception of abstraction. Their approach goes beyond Piaget's idea of empirical abstraction, as their understanding of abstraction accounts for the similarities of the underlying structures rather than the superficial (or external) characteristics of objects, as Piaget did. While Mitchelmore and White (2007) considered physical objects, Scheiner (2016) described a framework of a kind of abstraction, namely *structural abstraction*, that takes place on mental objects, and, even more important, considers *complementarity* of diverse features of mathematical objects instead of their similarity. The notion of structural abstraction has been introduced by Tall (see 2013) as a form of long term development in mathematical thinking with a focus on the properties of objects. Scheiner (2016) and Scheiner and Pinto (2014) further elaborated Tall's notion of structural abstraction to draw out the cognitive architecture of this kind of abstraction, accounting for both an objects-structure perspective and a knowledge-structure perspective. The data of a previous study (Pinto, 1998) was revisited, offering in the present paper a context for insights into students' sense-making of formal mathematics through the lens of the structural abstraction framework. Reinterpreting the data resulted in, and still contributes to, an evolving framework that may serve as a potentially useful tool in analyzing cognitive processes in mathematics learning with students' particular sense-making strategies that have not been captured by abstraction-from-actions approaches.

In this paper, we build upon previous research using the evolving framework of structural abstraction in providing insights in students' mathematical concept construction compatible with their sense-making strategy of 'giving meaning' (Scheiner, 2016; Scheiner & Pinto, 2014). Particularly, we take the revision of a case study of a student, called Chris, as a point of reference (Scheiner & Pinto, 2014) – a first-year undergraduate mathematics student, who "consistently understood [the formal concept] by just reconstructing it from the concept image" (Pinto, 1998). The object of consideration in this paper is another student, called Colin, who – similar to Chris – 'gave meaning' to the formal content. We begin this paper by sketching the structural abstraction framework and the research methodology of our project. The selected instances from Colin's case do not only highlight the analytical power of the structural abstraction framework but also indicate profitable directions for its advancement. It is important to note that the overall agenda in developing a theoretical framework of structural abstraction is not to challenge or explain ideas presented in an original work or to contrast and compete with recent approaches in mathematics education but to theorize about, to provide deeper meaning to older ideas, and to take them forward in ways not conceived yet.

THEORETICAL BACKGROUND

Structural abstraction is proposed as embedded in a cognitive architecture that takes place both on the objects-structures and on the knowledge-structures. It has a dual nature: (1) *complementarizing* the meaningful aspects and the structure underlying specific objects falling under a particular mathematical concept, and (2) promoting the growth of coherent and complex knowledge structures through *restructuring* of the knowledge system gained through the former process.

From the objects-structure perspective, we assume that the meaning of a concept is almost always contained in a unity of meaningful components of a variety of specific objects that fall under the particular concept. For the (socially constructed) meaning of a mathematical concept we draw on Frege's (1892) observation that the meaning is not directly accessible through the concept itself but through objects that fall under the concept. In this sense, we cannot take as absolute the 'complete construction' of the meaning of the concept. Rather than trying to draw a sharp line between whether an individual has (or has not) constructed the whole meaning of a mathematical concept, or to elaborate stages of objects-structure development, we pay particular attention to partial constructions of the concept that students develop, and how they make use of them in constructing new knowledge. In our view of the structural abstraction framework, a concretizing process is demanded to particularize meaningful components and the underlying structure of an object falling under the mathematical concept. Concretizing may occur through contextualization that is, placing object(s) in different specific contexts. Structural abstraction, then, means (mentally) structuring aspects and the underlying structure of these specific objects. In contrast to an empiricist view whose conceptual unity relies on the commonality of elements, it is the interrelatedness of diverse elements that creates unity. Thus, the core mechanism of structural abstraction is complementarizing rather than seeking for similarity. In addition, we suggest that, in the complementarizing process, a representation may be developed that is used generically for several other instances, and, in doing so, may provide a theoretical structure in constructing the meaningful components of the objects. Here we draw on Yopp and Ely's (2016) insightful contribution indicating that what makes an example generic has not only to do with whether the example is a carrier of the general but also with the actions performed on it – a lesson that Balacheff (1988) tried to teach long ago.

For students who 'give meaning' such 'representations of' are used generically as 'representations for' sense-making in mathematics. This shift from establishing a representation of a concept to using this representation generically for constructing and reconstructing the concept in new contexts, could be described in terms of shifting from a 'model of' to a 'model for' (Streefland, 1985). Models are, in this sense, intermediate in abstractness between 'the abstract' and 'the concrete'. This means that in the beginning of a learning process a model is constituted that supports the 'ascending from the abstract to the concrete' as described by Davydov (see 1972/1990). Davydov's strategy of ascending from the abstract to the concrete draws the transition from the general to the particular in the sense that learners initially seek out a primary general structure, and, in further progress, deduce multiple particular features of objects using that structure as their mainstay. The crucial aspect in this approach is Ilyenkov's (1982) observation that "the concrete is realized in thinking through the abstract" (p. 37). The key feature within the objects-structure perspective, however, lays in the idea that specific objects falling under a particular concept mutually complement each other, so that the abstractness of each of them, taken separately, is overcome. In this sense, and in line with a dialectical perspective described by Ilyenkov (1982) but different from empiricist approaches, structural abstraction is a movement towards complementarity of diverse aspects that creates conceptual unity among objects.

From the knowledge-structure perspective, we take the view that knowledge is a complex system of many kinds of knowledge elements and structures. Structural abstraction implies a process of restructuring and expanding the knowledge system, consisting of such 'pieces of knowledge' that have been constructed through the processes described above. The cognitive function of structural abstraction is to facilitate the assembly of more complex knowledge structures. The guiding philosophy of this approach is rooted in the assumption that learners acquire mathematical concepts initially on their backgrounds of existing domain-specific conceptual knowledge through progressive integration of previous concept images and/or by the insertion of a new discourse alongside existing concept images.

The reanalysis of empirical data gained from Pinto's (1998) study has shown that students, who give meaning, build a representation of the concept and, at the same time, use it generically for reconstructing the concept in other contexts – such as in verbal recovering the formal definition. The analysis also showed that students generically used representations of the concept to build pieces of knowledge. To put it in other words, the representations are actively taken as representations for producing new knowledge and sense-making of mathematics. This mental shift from 'representations of' to 'representations for' may indicate a degree of awareness of the meaningful components and a level of complexity of the knowledge system (Scheiner & Pinto, 2014). In this paper, we discuss one students' non-linear knowing and learning development of the limit concept of a sequence.

RESEARCH PURPOSE

The purpose of the paper is twofold: (a) refining and extending the theoretical framework through paying particular attention to eventually unsettled issues about structural abstraction, and (b) providing further insights in its potential power for the analysis of an individual's partial construction of the limit concept of a sequence, consistent with his sense-making strategy. In doing so, we focus on those aspects of the learning phenomena that are illuminated by using the structural abstraction framework (and that have not been noticed before). Thus, the framework functions both as a tool for research and as an object of research, a distinction already made by

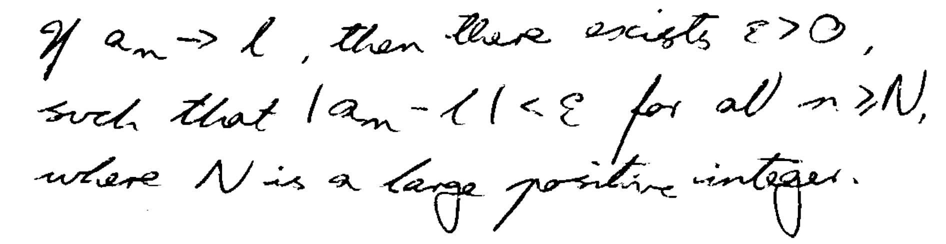
Assude, Boero, Herbst, Lerman, and Radford (2008). Our agenda is driven by re-examining an earlier study (Pinto, 1998) that identified a sense-making strategy of formal mathematics that has not fully been captured by abstraction-from-actions approaches in the literature on knowing and learning mathematics. The original data were collected taking an inductive approach throughout two academic terms during students' first-year at a university in England, through classroom observation field notes and transcriptions of semi-structural individual interviews. Interviews took place every two weeks with eleven students in total. A cross-sectional analysis of three pairs of students resulted in an identification of two prototypical sense-making strategies: 'extracting meaning' and 'giving meaning'.

> "*Extracting meaning* involves working within the content, routinizing it, using it, and building its meaning as a formal construct. *Giving meaning* means taking one's personal concept imagery as a starting point to build new knowledge." (Pinto, 1998, pp. 298-299)

The latter strategy is the object of our study. In this paper, we selected instances from the available data of the case study of a particular student, called Colin.

SELECTED INSTANCES FROM A CASE STUDY

At the beginning of his first course on real analysis, Colin expressed, in his first interview, the formal definition of the limit of a sequence as follows:


If $a_n \to l$, then there exists $\varepsilon > 0$, such that $|a_n - l| < \varepsilon$ for all $n \geq N$, where N is a large positive integer.

(Pinto, 1998, p. 201)

His partial reconstruction of the formal concept definition of limit is a productive formulation (in a sense that it works in various contexts) of a property of a convergent sequence. His sense-making is coherent with his written definition:

> Umm ... it means that the difference between … umm … the terms in sequence a_n and the limit gets very very small indeed and it gets below a certain umm pre-determined value. […]. Err ... yes, after you go far enough out in the sequence. (Colin, first interview)
>
> (Pinto, 1998, p. 203)

A dynamic view of a sequence as a process, implicit in evoked images such as 'you may go far enough out', and of the limit concept in terms of 'getting very small' are both indicated. Images such as 'arbitrarily small quantities', or 'infinitesimals', which are common in secondary school learning settings, are recalled with the use of the dynamic language of 'gets very very small indeed'. This is consistent with results in Martinez-Planell, Gonzalez, DiCristina, and Acevedo (2012) on students' understanding of series. The authors focused on whether students were seeing series as a process without an end or as a sequence of partial sums, as stated by definition; and referred to Arnon et al.'s (2014) APOS theory to respond how students may construct the notion, by considering a distinction amongst the understandings of the concept of a sequence as a list of numbers or as a function defined in natural numbers (McDonald, Mathews, & Strobel, 2000). Martinez-Planell et al. (2012) concluded that even after formal training, students often think of sequences and series as an infinite unending process, and evoke dynamical aspects, as Colin did.

We approach the phenomena from the perspective of the structural abstraction framework and understand that, in line with his personal concept definition, the result of Colin's contextualizing processes resulted in a representation of the limit of a sequence as that of a descending curve (see Fig.1):

> ... umm, [I] sort of imagine the curve just coming down like this and dipping below a point which is epsilon... and this would be *N*. So as soon as they dip below this point then ... the terms bigger than this [pointing from *N* to the right] tend to a certain limit, if you make this small enough [pointing to the value of epsilon]. (Colin, first interview)
>
> (Pinto, 1998, p. 202).

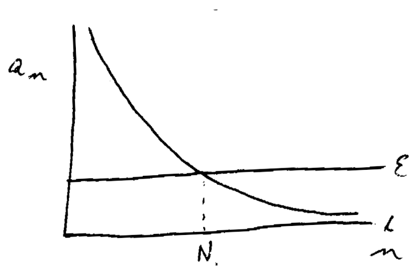


Fig. 1: Colin's first picture

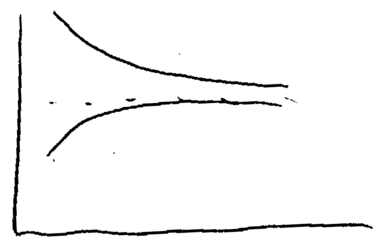


Fig. 2: Colin's second picture

Colin accentuated the image of a decreasing sequence by saying "as soon as they dip below the point then". The second picture he drew (see Fig. 2) is based on the idea that "convergence could happen from above and below". In other words, it seems that he evoked images of a convergent sequence identifying it with monotonic ones. He was able to explore his representations dynamically, interpreting and exploring the actions involved in his written definition relying on partial constructions that are specific and productive to some familiar contexts. In this light, it seems that Colin has interpreted (in the sense of Piaget's notion of assimilation) new concepts in terms of his prior knowledge:

> Umm ... in A' level we used to … umm … plot sequences and generally you might get a sequence like this and ... *it would tend down* to a value or something. … the little bit I had done at A-level I just sort of settled into it quite well… (Colin, first interview)
>
> (Pinto, 1998, p. 203)

Thus, from the objects-structure perspective, Colin's partial reconstruction of a convergent sequence still related to a descending function or sequence, in a context where a formal discourse is inserted. From the knowledge-structure perspective, his *representation of* a convergent sequence was generically used as a *representation for* constructing knowledge, as those students who 'give meaning' did. On the one hand, his representation of a convergent sequence was productive, as many times he sensed when results and claims were true. On the other hand, many statements became self-evident for Colin while his earlier mathematical discourse still was not recontextualized within the formal experience, as when he was asked to prove:

> If $a_n \to 1$, prove that there exists $N \in IN$ such that $a_n > \frac{3}{4}$ for all $n > IN$.
>
> Colin said: 'It seemed to be a silly question that … if a_n tends to 1 then if you question when a_n is greater than $\frac{3}{4}$ …this is a bound, it seems … I don't know why.' (Colin, third interview)
>
> (Pinto, 1998, p. 221)

Colin's representation of a convergent sequence and its limit, which was coherent with his sense-making and his written definition, is a potential conflict factor (Tall & Vinner, 1991) concerning its use and the formal discourse. It did not enable him to produce a formal proof. Here, other than seeing the formal content as demanding, it is its complementary aspect that matters. Colin eventually noticed the new discourse introduced by the formalism as increasingly conflicting with his sense-making of the theory. In many occasions he ignored it and simply added it as an information:

> There are certain things that ... I think they're okay and I just learn that, it's sort of that's defined to be that ... ... (Colin, seventh interview)
>
> (Pinto, 1998, p. 205)

In synthesis, in many contexts and situations, students may activate the various partial constructions productively. Such an attitude could be common; but in Colin's case, various issues related to recontextualization of his concept image seem to miss.

DISCUSSION

In this paper, we presented data showing that a student, called Colin, built partial constructions of a convergent sequence that he used as representations of the concept. Such use was productive to particular contexts, but remained unproductive in others; for instance, to deal with formal mathematics. Colin could perceive that a statement is true, based on the properties of the concept he observed and concretized in a representation that he used generically (see Yopp & Elly, 2016), as a representation for building knowledge. Using his representation of the limit concept as a 'definition' Colin was able to evoke formal results, although he was unable to make deductions. Colin's awareness of the formal requirements in the new context at university was not immediate. His description of the natural flow of his transition from school to university, expressed during his first interview, indicates that he did not perceive that the concretized knowledge he learned at school and the formal context at university were already in conflict. As the course progressed, he was gradually conscious of conflicting aspects in his understanding; though he added new knowledge as information rather than (re-)structuring the prior mathematical experience. There are students whose sense-making of the mathematics is detached from their learning of the institutional knowledge. They deal with those as if sense-making and institutional knowledge were compartmentalized knowledge structures (see Vinner, 1991, p. 70). What strikes us in the selected instances of Colin's case study was the cohesion in his sense-making and in learning the formal mathematics concept. Coherence amongst students' sense-making and their (re-)construction of the formal content has been proven to be a central characteristic of those students who 'give meaning'.

From the objects-structures perspective of the structural abstraction framework, the aspect of 'complementarizing' meaningful components reflects the idea that whether an individual has 'grasped' the meaning of a concept can only be considered in specific contexts. This makes clear that "the subjective nature of understanding […] is not […] an all-or-nothing state" (Skemp, 1986, p. 43). A comparison with Chris' case (Scheiner & Pinto, 2014), another student who 'gave meaning', shows that although Chris did not 'have' all relevant meaningful components at hand, he was able – using his 'generic representation' – to generate some of them at need. The growing complexity of his representation of the convergence of a sequence, gradually constructed in particular settings, served as a representation for reconstructing and recontextualizing the limit concept in the formal context. We argue that Colin's understanding of a convergent sequence must increase in complexity and complementarity, which could be achieved through contextualizing as well as integrating various constructions; the latter may be promoted by the insertion of new mathematical discourses alongside earlier concept images.

From the knowledge-structures perspective, structural abstraction is a process of restructuring the 'pieces of knowledge' constructed through contextualizing and complementarizing. In using the structure of the representation, some meaningful components of the concept may be productively activated in diverse contexts. Such use may even allow to generate new knowledge pieces. In both cases, Chris and Colin, a shift from a representation of (the convergente sequence) to a representation for generating knowledge can be documented. The shift does not result in knowledge restructuring per se, as we could identify in Colin's case. On the other hand, Chris' case suggested that even a 'representation for' may be complementarized by new knowledge elements, and such a process becomes recursive (Scheiner & Pinto, 2014).

CONCLUDING REMARKS

The structural abstraction framework takes the view that knowledge is an evolving, complex, and dynamic system of many kinds of knowledge elements and structures. Abstraction is seen as a movement across levels of complementarity and complexity (Scheiner, 2016). The case study in Scheiner and Pinto (2014) and the one provided in this paper raise directions for advancing our understanding of structural abstraction. Both cases reveal (1) a *cohesion* amongst their sense-making strategy of giving meaning, and (2) a *generic use* of their constructed representations to reconstruct the limit concept in other and new contexts. Contrasting the two cases shows that the two students differed in the degree of complementarity and complexity of the representations used. Chris' representation could be considered as being generic in terms of being a carrier of the general (Mason & Pimm, 1984) that he used to reconstruct meaningful components at need. Colin's representation did not allow him to do so – maybe due to its degree of complexity and complementarity. Other questions to be addressed are raised by the use of representations in knowledge structuring – as a tool to reconstruct knowledge, as Chris did, or as an object in place of the definitions, maintaining the earlier mathematical discourse, as in Colin's case.

REFERENCES

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education*. New York, NY: Springer.

Assude, T., Boero, P., Herbst, P., Lerman, S., & Radford, L. (2008). The notions and roles of theory in mathematics education research. In *11th International Congress on Mathematical Education* (pp. 6-13). Monterrey, Mexico: ICMI.

Balacheff, N. (1988). A study of students' proving processes at the junior high school level. In I. Wirszup & R. Streit (Eds.), *Proceedings of the Second UCSMP International Conference on Mathematics Education* (pp. 284-297). Reston: NCTM.

Davydov, V. V. (1972/1990). *Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula*. Reston, VA: NCTM.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Dordrecht: Kluwer.

Frege, G. (1892). Über Begriff und Gegenstand (On concept and object). *Vierteljahresschrift für wissenschaftliche Philosophie*, *16*, 192-205.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: a proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, *26*(2), 115–141.

Hershkowitz, R., Schwarz, B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education*, *32*, 195-222.

Ilyenkov, E. V. (1982). *The dialectics of the abstract and the concrete in Marx's Capital*. Moscow: Progress.

McDonald, M. A., Mathews, D., & Strobel, K. (2000). Understanding sequences: A tale of two objects. In E. Dubinsky, J. J. Kaput, & A. H. Schoenfeld (Eds.), *Research in Collegiate Mathematics Education* (Volume IV, pp. 77-102). Washington, D.C.: Mathematical Association of America.

Martínez-Planell, R., Gonzalez, A. C., DiCristina, G., & Acevedo, V. (2012), Students' conception of an infinite series. *Educational Studies in Mathematics, Educational Studies in Mathematics, 81*(2), 235-249.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics*, *15*(3), 277-289.

Mitchelmore, M. C., & White, P. (2007). Abstraction in mathematics learning. *Mathematics Education Research Journal*, *19*(2), 1-9.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Dordrecht, The Netherlands: Kluwer.

Piaget, J. (1977/2001). *Studies in reflecting abstraction* (*Recherches sur l'abstraction réfléchissante*) (R. L. Campbell, Trans.). Philadelphia: Psychology Press.

Pinto, M. M. F. (1998). *Students' understanding of real analysis*. Coventry, UK: University of Warwick.

Scheiner, T. (2016). New light on old horizon: Constructing mathematical concepts, underlying abstraction processes, and sense making strategies. *Educational Studies in Mathematics*, *91*(2), 165-183.

Scheiner, T., & Pinto, M. M. F. (2014). Cognitive processes underlying mathematical concept construction: The missing process of structural abstraction. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 5, pp. 105-112). Vancouver, Canada: PME.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1-36.

Skemp, R. R. (1986). *The psychology of learning mathematics*. (Second edition, first published 1971). London: Penguin Group.

Streefland, L. (1985). Wiskunde als activiteit en de realiteit als bron [Mathematics as an activity. and the reality as a source]. *Tijdschrift voor Nederlands Wiskundeonderwijs (Nieuwe Wiskrant)*, *5*(1), 60-67.

Tall, D. O. (2013). *How humans learn to think mathematically. Exploring the three worlds of mathematics*. Cambridge, UK: Cambridge University Press.

Tall, D. O. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limit and continuity. *Educational Studies in Mathematics*, *12*(2), 151-169.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81). Dordrecht: Kluwer.

Yopp, D. A., & Ely, R. (2016). When does an argument use a generic example? *Educational Studies in Mathematics*, *91*(1), 37-53.