A model from signal analysis to design linear algebra activities

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In this paper we present a didactical activity based on modelling an engineering problem known as Blind Source Separation (BSS) and the results of its implementation in a linear algebra course in a Mexican university. The problem had previously been analysed from an institutional point of view, carrying out the notion of the matrix map T(x)=Ax. In the frame of APOS theory we propose a genetic decomposition for this concept, in order to analyze students' constructions related to it, and at the same time, using the BSS context as a reference to connect mathematical constructions with a real life situation.

Keywords: modelling, engineering, signals, BSS, linear algebra.

INTRODUCTION

In this paper we present a didactical activity based on a real engineering problem known as *Blind Source Separation,* and the results of its implementation in two groups in a Mexican university. In previous research The Anthropological Theory of didactics (ATD) (Chevallard, 1999) was used to analyze the mathematical concepts embedded in this engineering context from an institutional point of view. As a result, we identified that the notion of linear transformation and its matrix formulation, play an important role in modelling this problem. We also reported that some other notions such as that of signal and sampling are important concepts in the training of engineering students and that they can be introduced in the teaching of mathematics for engineering students.

We consider that if these notions are introduced early in the mathematics curriculum for engineering students, they can become a tool for reducing the gap between what students learn in mathematics courses and the way they should apply this knowledge to solve real problems in the context of their profession; an issue that has been widely studied (Kent y Noss, 2002).

In order to achieve this goal, we selected the notion of a matrix linear mapping and its inverse in the context of signal separation as the starting point to design a didactic activity for an introductory linear algebra course for engineering students.

The analysis presented in (Vázquez, Romo-Vázquez & Trigueros, 2015) was intended to study the context known as Blind Source Separation (BSS), a problem from Signal Analysis that was first established in order to study motion decoding in vertebrates (Comon & Jutten, 2010).

In that study, using the notions of praxeology and institution from ATD we distinguished, in first place, the mathematical tasks and techniques involved in formulating and solving BSS when the institution of reference is Signal Analysis. Then we identified mixed praxeologies considering both mathematical knowledge and professional practice that could be didactically transposed to an introductory Linear Algebra course.

In general terms, BSS is about separate information that is measured in form of signals. The problem consist in retrieving n source signals $\mathbf{s}=(s_1,s_2,\ldots,s_n)$ that are mixed under a linear model $A\mathbf{s}=\mathbf{x}$, when only the observed signals $\mathbf{x}$ are known. The research on separation methods has generated a whole research area known as Independent Component Analysis (Comon & Jutten, 2010). There is a rich variety of applications on BSS: studying the brain information obtained with electroencephalograms and other biomedical signals, processing satellite images, as well as radioastronomy, sound, GPS or interfered signals, are a few examples.

As a result of the analysis we found that the notion of signal (defined in engineering courses as a function) can elicit students' reflection on the relationship between the type of functions (and their graphs) commonly studied in a Calculus course and those needed in signal analysis, and differences in their graphical representation. In particular, the sampling of a signal relates the concept of *function* with that of *vector*. The mathematical model for BSS, which works as an input-output system in the form of a linear system of equations, makes clear, from the start, the need to relate concepts such as linear system, linear transformation and the map $x \rightarrow Ax$.

A rich body of literature exists about students' difficulties when learning Linear Algebra concepts and also about the use of modelling in the learning of concepts in this discipline, particularly using APOS theory (Trigueros, Oktaç & Manzanero, 2007), (Possani, Trigueros, Preciado & Lozano, 2010). Also, the map $T:R^m \rightarrow R^n$, defined by $T(\mathbf{x})=A\mathbf{x}$, can be regarded as an extension of the notion of a function of several variables which has been studied by Martinez-Planell & Trigueros, 2010) using the same theoretical framework. However, there are important differences between these two types of functions as $A\mathbf{x}$ also entails the matrix-vector product. Taking results of previous studies into account and considering that the use of a non-mathematical problem can help students to make sense of these new functions and to abstract the main mathematical ideas involved in their construction, we designed a modelling activity based on BSS context. The goal of this part of our research project was to add an analysis of the constructions needed in the learning of the concepts of linear transformation and its inverse using APOS Theory to results identified in the praxeological analysis of the context to design a modelling activity, and a set of activities to introduce students to both, the ideas related to BSS as an engineering problem and matrix transformations.

Our research questions were: What are the constructions involved in relating transformations with matrices? What are the constructions needed in the learning of inverse transformation and inverse matrix? Does the use of an extra-mathematical situation play a role in favouring those constructions?

THEORETICAL FRAMEWORK

APOS Theory was used as a framework to study students' constructions (Arnon, et al. 2014). It intends both to model the way students learn advanced mathematical topics in order to design teaching sequences that can proved to be effective in terms of students' learning, and to analyze the knowledge that students display when solving a specific activity at a particular moment in time. When using APOS theory researchers take into consideration students' previous knowledge. The application of APOS theory to describe particular constructions by students requires researchers to develop a genetic decomposition (GD) – a description of specific mental constructions a person may make in the process of understanding mathematical concepts and their relations. Students work collaboratively in groups discussing and responding to specific tasks contained in the pre-designed activities. Different kinds of activities, which have particular aims, are carefully developed based on the GD. In some activities students need to perform actions on objects and reflect on them. Other tasks have as a goal to interiorize those actions into processes. Reflection on how and why they work helps students abstract their main characteristics, take control over them and flexibly use them. There are also tasks, which intend to make students reflect on the process and be aware of it as a totality so that they can apply new actions to it. When this happens the process is encapsulated into an object. Tasks are also designed to help students be aware of the relations among actions processes and objects and also on the relations to other concepts. The theory refers to these collections as schemas. Schemas evolve as new relations between new and previous actions, processes, objects and other schemata are constructed and reconstructed.

METHODOLOGY

For the design we proposed a GD for the matrix map $T(\mathbf{x})=A\mathbf{x}$ and its inverse: The construction of the map T involves the interiorization of actions of evaluation of a map on different points in a vector space. This process can be encapsulated into an object when properties of the map are studied. A schema for transformations as functions involves the construction of the domain and range sets as objects and the construction of a relation where the process of real valued function is coordinated with the transformation process to consider both as functions that differ in their domain and range. The transformation process is coordinated to the product of matrix and vector process into a new process where specific transformations can be described in terms of this product. Actions on matrices and vectors and on interchanging the order of factors are involved in the construction of a process construction of $A\mathbf{x}$. The coordination of the function schema, euclidian space R^n, and the process for the product $A\mathbf{x}$, results in the coordination of the vector resulting from the product $A\mathbf{x}$ as a vector image of T. The construction of solution set of a linear system of equations as an object is necessary to pose and solve questions about the possibility to find the preimage of a vector in the range of T, and to find an inverse of

A and the inverse of a transformation as objects. Actions on the map T and its inverse to validate the rule $TT^{-1}=I$ and determine its unicity on the domain of T help in encapsulation of these transformations as objects. The didactical activity we propose in this paper intends to foster these constructions, using BSS as a modeling frame of reference.

A first design of the modeling of the source separation problem in an audio context was previously tested with a pilot group. The designed activity was probed with 24 students in a linear algebra group. The analysis of the results of this experience, and in particular, the modeling process constitutes the focus of this paper. Five sessions of two hours each were used in this experience. This group had previously worked with activities based on APOS theory to construct a linear system of equations schema. The teacher had also introduced Gauss-Jordan elimination method, and the product of a matrix and a vector, by means of dot product. At the time we tested the activity, matrix product, inverse matrix and linear transformation had not been introduced. Students worked in teams of six, first discussing within their team and later exposing their conclusions to the whole group. In the first two sessions they worked in the construction of a mathematical model for BSS and linear maps $\mathbf{x} \rightarrow A\mathbf{x}$. Next the experience focused on the inverse of the map. The last two sessions intended to introduce the inverse of a transformation and of a matrix through activities based on the GD. All the sessions were observed by the researchers and video-recorded. In the next section we present some representative activities from the design used in the classroom, and results from the analysis of the obtained data.

DISCUSSION AND RESULTS

First part: the modelling activity

As an initial problematic situation, we posed the problem*: Let's suppose we are spying an important meeting. We have put some recorders in some places of the office where the meeting is hold. We also have a map of the seats of the important people we want to spy, but we cannot see them. After the meeting, the only thing we have are the recordings of the conversation -most of the time there are more tan two people talking at the same time- and a map of the location of the recorders*. (We in fact reproduced the sound of several mixed voices in several recordings, placed in different locations of the classroom). *How can we indentify, with only this information who is talking and where is she/he seated?*

The purpose of this problem situation was to introduce a context where a mathematical model could be developed to find a response to the problem's question. As it has been recommended in previous modelling based research, one of the first activities when introducing a modelling task is the recognition of relevant variables in the situation. We then asked the group (working on teams of six students) for the types of variables they could identify. The constructions students need to make to propose variables are related to mathematical and non mathematical schemas, such as physics of sound, space, the mechanism of recording, among others. The distance between sources and recorders, as well as the tone of voice of the speakers were variables that students considered as crucial. Other variables proposed were the noise, the echo at the office, or voice fluctuations. The ability to define the main variables in an open modelling situation is mentioned in (Hamilton, Lesh & Lester, 2008). The next dialogue shows how decisions when considering variables, which is an important competence in engineering education, were taken by students in a team:

Student 1: We think that the shape of the office is important because sound bounces and makes it more difficult to distinguish the speakers. And, it is possible that, when sound wave reaches the wall sounds can cancel each other out.

Student 2: I disagree. If you only have the recording and a 2D map, you cannot take into account the shape of the room. I know it is important but you have to drop it out.

Tutor: So we drop out that variable from our list?

Students 1 and 2: Yes.

In order to establish the instantaneous linear mixture model for BSS the teacher and the researchers proposed a tool named "configuration". A configuration is a two-dimensional representation of the location of voices (called sources) and recorders (observations). We presented four different configurations (varying the number of sources, observations and distances between them) and asked the question: *in which of these configurations would it be easier for the spy to solve the problem?* Students considered that if number of sources exceeds by far the number of observations, the sound would be "too mixed" in each recorder. They also discussed that, on the other hand, many observations and few sources entails "too much" information and makes the task of separating it difficult. Most of them decided that the best configuration to be able to separate sounds was the one shown in diagram C. Later on, when the matrix form of the model appeared, students' were able to relate these conclusions to the size of a matrix representing the configuration and the possibility of finding its inverse. During students' discussion the issue of the speaker's voice tone also appeared. Students were not sure how to deal with it, so the teacher helped them to consider that the speaker's voice at an instant could be simplified as a pure tone $y=\sin(2\pi wt)$. She suggested students' to use an online tool in order to produce pure tones with different frequencies, w. They performed actions of changing frequency and simulated different distances between sound (source) and receptor (ear). From this exploration students were able to construct a mathematical model of the sound received at an instant by a recorder. This model related sound amplitude to distance from the source as an inverse proportionality. Finally, students were able to develop a linear mathematical model for each observation as a linear combination of pure tones. As a closing discussion, together with students, the teacher proposed a common notation for a set of instantaneous sources as a vector $\mathbf{s}=(s_1,s_2,\dots,s_n)$ and $\mathbf{x}=(x_1,\dots,x_m)$ for observations, where each s_i corresponds to a pure tone. She also named **s** the input and **x** the output of the mixing system.

Part two: the construction of matrix mapping T(x)=Ax

The goal of the second part of the experience was to investigate the constructions needed to relate the notion of the particular map T(**x**) to an input-output system, its matrix form, and the system of linear equations associated for each **b**=T(**x**).

The signal configuration tool was used to probe students' constructions on domain and range of a map. They were given the diagram below and asked:

a) Consider the mixing transformation that maps sources to observations in configurations A, B and C. What is the domain and range of each of them?
b) Suppose that sources in diagram C correspond to three pure tones with frequencies 440Hz, 660Hz and 880Hz, respectively. What is the image under the mixing map of the sources at instant t=2 seconds?
c) Write the mixing matrix for each configuration.

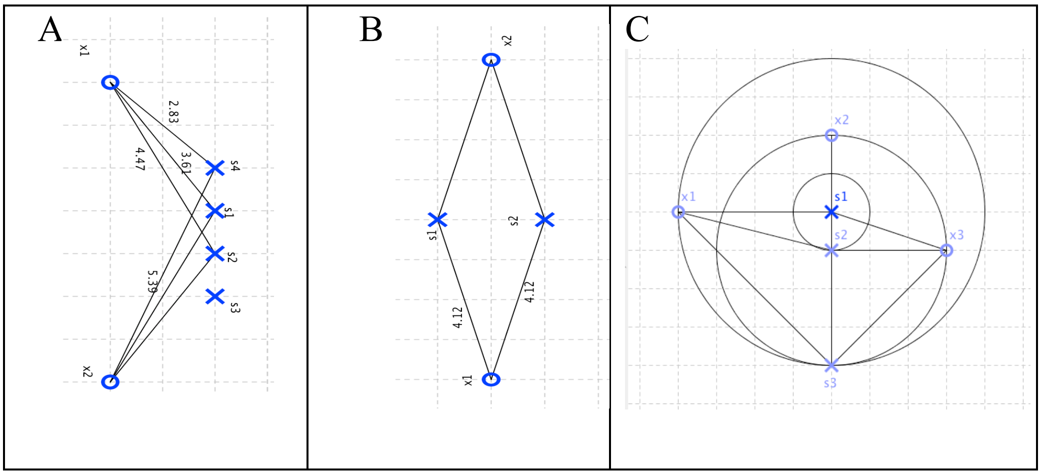

Figure 1: Three signal configurations to explore the mixing map T(x)=Ax

The first question probes students' constructions related to domain and range of a map. The construction of a consistent schema of these concepts requires a previously constructed schema of function (Martínez-Planell & Trigueros, 2010). As the mixing map relates vectors in R^n (where n is the number of sources) to vectors in R^m (m the number of observations) there is not a geometric representation of domain or range for n>3 (see diagram A). Students were asked to assign a vector **s** to a vector with the purpose of helping them reflect on their actions and interiorize them into a vector function process. Some students struggled to associate domain and range with Euclidean vector spaces; it was necessary to recall the representation of signals as their value at some instant, for them to recognize each of the sources as a real number, and therefore, an element in the domain as a vector formed by n values of those sources. Once this was done, students easily identified R^4 as the domain and R^2 as the range of the mixing map in figure A; they were able to find on their own the domain and range for transformations depicted in figures B and C. They were able, as well, to relate to the previous work with the model and to conclude that separation was easier when the dimension of both domain and range are the same. Furthermore, students coordinated this process to a previously constructed process for domain of single-valued functions into a domain process as the set where both types of functions are defined; necessary to answer question a). The dialogue of the teacher with Student 3 shows the difficulties found by students who have not constructed function as a process:

Student 3: I know that domain is where variables are ok but here… I can´t see here if there are problems with the sources.

Tutor: What do you mean when you say that they are ok?

Student 3: Yes, for example: to obtain the domain of a function I solve the inequality or look at points where the denominator equals zero. But I can´t see that in this example, because its sin(x).

Student 3 showed an action conception of a domain. She had previously worked with real functions of a single variable, and their analytic expressions, solving inequalities or indeterminacies. Regarding the definition of domain as the set where the function is defined, other students included words from the context of the problem:

Student 4: The domain depends on the voices of speakers.

Tutor: How can that be?

Student 4: Umh… I am not sure, but I think that only the voices of speakers are in the domain, the domain cannot include all possible tones.

Tutor: Does that means that the variables for the mixing map are the pure tones?

Student 4: Yes, I think so.

Student 5: No. I don´t think so. The domain is R…each source is a function whose domain is R because sinus function is continuous on all R.

Student 5's answer shows he has not interiorized yet the notion of variable in the domain beyond the context of single-valued real functions; more actions on different types of functions need to be performed so that students can interiorize them into a process construction of domain. The explanation given by Student 4 shows an interaction between her schema for the external world related to the problem situation and her domain schema. Her schema for domain, however, only contains processes related to elements where it is "useful" to evaluate the function. Some of these obstacles were also found in the case of the range of the function. Once domain and range of the mixing map were institutionalized in a whole group discussion, students went on to solve item b) which they were able to answer pretty fast. We observed that they progressed better when they had a specific value to obtain source vector **s** as an arrange of real numbers, which evidences that most of them needed to do more work in order to interiorize domain and range as processes. Most of them reflected a coherent construction of the image under a function of an element in the domain, as they did actions of calculating each value $s_i(2)$, they did also the actions of combining them linearly and grouping them to form the image vector $\mathbf{x}=T(\mathbf{s}(2))$.

Finally, item c) intended to probe if students' previously constructed structures about matrices and vectors enabled them to recognize the product of a matrix A and a vector **s** in the model for the transformation of sources and observations they were developing. We intended to observe if they were able to relate these constructions to BSS contextual elements in order to explain, beyond mathematics, the need for A to have n columns if vector **s** is in R^n. Results obtained showed that effectively, most students had interiorized the matrix form of a system of equations into a process and could coordinate it with a process of coefficient matrix once the nxm linear system was identified in item b). Students related the size of the matrix A to the BSS context by observing that the number of columns of A must equal the number of sources, and the product results in m observations, so they concluded A has to have m rows by making reference to configurations in each case.

An interesting result emerged when students worked with a symmetrical configuration of 2 sources and 3 observations, where it was asked if –due to symmetry- the information received by x_1 and x_2 was the same. Some of the students answered this question starting by writing the system, then its coefficient matrix and gave a clear explanation in terms of linear dependence of the rows of A, showing they were able to use linear dependence as an object and relate it to the problem.

Part three: A genetic decomposition for A^{-1}

The next step in working with this model was to recognize the inverse matrix as a tool to solve the separating sources problem. It is worth to mention that the design simplifies the conditions of the BSS problem by posing a non-blind problem, where the matrix, if not given, can be deduced.

Students recognized during work with the problem the usefulness of having an inverse map for T(**x**)=A**x** in order to separate sources. A discussion on how to find the inverse map was opened. As the mixture map is represented by a matrix, students assumed the fact that the inverse would also have a same sized matrix representation. This fact was not considered initially in the genetic decomposition and should be part of a refinement.

We need to recall now that matrix product was not yet constructed by students. This was a decision of the researchers because product matrix, seen from a linear transformation perspective corresponds to the composition of maps. Yet, in BSS a composition of mixtures makes little or no sense. A 2x2 matrix representing a mixture transformation was presented to students and the corresponding mixtures were shown in Geogebra (where the sources were hidden); specific values of the mixtures were thus available, and the problem of determining the corresponding input vector, given a specific "output" vector **x** chosen by students was posed. Students who showed a process construction of linear system of equations, posed the corresponding system and solved it for two variables. They asked for "more information" and used the mixtures to obtain it. After some work, they arrived to the inverse matrix of A. Other students stated that it was impossible to obtain the input, or that information given was not enough. These responses evidenced they had constructed action conceptions of linear system and its solution set, as they only referred to the first data given and were able to solve only that specific system. Later, students were asked to perform the same actions on different matrices and different output vectors, selected by researchers in order to explore the conditions for the existence of an inverse transformation or the inverse of matrix A. Some students who demonstrated a process or an object construction of linear system were able to relate the existence of A^{-1} with the linear independence of the rows of A and a suitable number of linearly independent output vectors. Students who showed an action conception were not able to find these general conditions but were able to solve exercises by doing actions and obtaining a conclusion for each given matrix. The following part of the activity was designed to help students construct the property that if the output vectors given were those of the standard basis for R^n, then the input vectors obtained were the columns of A^{-1}. Students did actions related to this and after reflecting and interiorizing them into a process they were able to construct by themselves the Gauss-Jordan algorithm to obtain the inverse of a matrix. In every case, the reflection on the validity of the argument arose, but, in general, students showed understanding of what they were doing. A small group struggled with the fact that solving the system $A\mathbf{x}=\mathbf{e}_1$ then $A\mathbf{x}=\mathbf{e}_2$, etc. separately was equivalent to reduce by Gaussian elimination A and I simultaneously, that is the augmented matrix (A|I). These students showed an action construction of the Gauss-Jordan elimination algorithm, in the sense that they can't recognize that variable **x** is just a label and doesn't modify the solutions of the system. We consider that this result can contribute to refine a genetic decomposition for the schema of linear system equations. The use of the visual interface (Geogebra) allowed the possibility to ask what would happen if, for a specific inverse matrix, calculated from output vectors, output data is changed. Students who had not an object construction of linear system doubted in deciding if the inverse changed. Through a whole group discussion the unicity of the inverse transformation was related to the BSS context, as the coefficients of the matrix, defined by the reciprocal of the distances between sources and observations are invariant. Further insight on this issue will be obtained from questions asked in a mid-term exam, and from semi-structured interviews with students to be conducted.

Finally, a brief presentation on the importance of the BSS problem was shown to students, together with some of its applications. Students proposed different situations where they thought separation of signals could be useful.

CONCLUSIONS

The modelling part of the design is suitable to trigger students' interest; in particular the work with pure tones elicits the use of different registers and makes the notion of vector a useful tool to represent a sampled signal. The BSS context broadens the possibilities to construct coherent schemas of domain, range and function. We propose a genetic decomposition for the matrix map and its inverse. Student's work on its construction showed that an object conception of a system of linear equations is necessary in order to relate the map with the product of a matrix and a vector, and to see it as an input-output system.

The design allowed students to construct an algorithm for finding the inverse of a matrix without using the matrix product and to explore conditions of existence of the inverse matrix related to linear independence of the rows of A. Research will continue by exploring constructions related to linearity. The adapted model of BSS used for the didactical design presented here, equivalent to $\mathbf{s}=A^{-1}\mathbf{x}$ seems to be a powerful tool to solve inverse problems in contexts beyond audio signals. Future work on this project will focus on the analysis of a final questionnaire answered by all the students in the group and of interviews conducted with selected students.

REFERENCES

Chevallard, Y. (1999) *L'analyse des pratiques enseignantes en théorie anthropologique du didactique.* Recherches en didactique des mathématiques 19(2), 221-266.

Kent, P., y Noss, R. (2002). *The mathematical components of engineering expertise: the relationship between doing and understanding mathematics*. I.E.E.E. Second annual symposium on engineering education.

Comon, P., y Jutten, C. (2010). *Handbook of blind source separation: Independent component analysis and applications*. Oxford:Academic Press

Vázquez, R., Romo-Vázquez, A., Trigueros, M., and Romo-Vázquez, R. (2015) *La separación ciega de fuentes: un puente entre el álgebra lineal y el análisis de señales.* Manuscript submitted for publication.

Trigueros, M., Oktac, A., y Manzanero, L. (2007). *Understanding of systems of equations in linear algebra.* En D. Pitta y G. Philippou (Eds.), Proceedings of the 5th congress of the european society for research in mathematics education (p. 2359-2368). Larnaca, Cyprus.

Possani, E., Trigueros, M., Preciado, J. G., y Lozano, M. D. (2010). *Use of models in the teaching of linear algebra.* Linear Algebra and its Applications, 432 (8), 21-25

Martínez-Planell, R., Trigueros, M. (2010) *Students' understanding of the general notion of a function of two variables.* In Educational Studies in Mathematics. An International Journal, Vol. 81, Núm. 3 pp. 365-384.

Hamilton, R., Lesh, R., y Lester, M. (2008*). Model eliciting activities as a bridge between engineering education research and mathematics education research. Advances in Engineering Education*, 1 (2), 1-25.

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A. , Roa Fuentes, S., Trigueros, M. and Weller, K. (2014) *APOS Theory. A framework for Research and Curriculum Development in Mathematics education.* Springer.