From ‘monumentalism’ to ‘questioning the world’: the case of Group Theory

Marianna Bosch[1], Josep Gascón[2], and Pedro Nicolás[3]

[1]IQS School of Management, Univ. Ramon Llull, Spain, marianna.@iqs.edu;
[2]Dt. Mathematics, Universitat Autònoma de Barcelona, Spain, gascon@mat.uab.cat;

[3]Pedro Nicolás, Dt. Didactics of Social Sciences and Mathematics, Universidad de Murcia, Spain, pedronz@um.es

One of the main research gestures within the Anthropological Theory of the Didactic (ATD) consists in questioning the mathematical knowledge that is at the core of teaching and learning processes. In the case of university education, and because the knowledge at stake is closely related to scholar knowledge, this questioning might seem less necessary and the tendency is to focus on the possible different ways of organizing its teaching. Through the example of Group Theory we illustrate the methodology of questioning proposed by ATD in two different moments of the evolution of this framework: a first one centred on the structure and dynamics of the different components of mathematical praxeologies; a second one more focused on the raison d’être and functionality of what is taught and learnt.

Keywords: anthropological theory of the didactic, group theory, praxeologies, study and research paths

INTRODUCTION

Together with school Mathematics at secondary level and teacher education, university education has been one of the focuses of our research team during the past 25 years. The reason is partially circumstantial. As members of Mathematics Departments at the university, we are close to our object of study and are even part of it as teachers, what makes empirical studies easier. In addition, facing university teaching difficulties first-hand obviously contributes to putting them at the centre of our research agenda.

During these years, the approach we use, the Anthropological Theory of the Didactic (ATD), has strongly evolved with the introduction of the notion of praxeology (Chevallard et al 1997, Bosch & Chevallard 1999). Looking back at our first investigations concerning a new teaching device called ‘Workshops of Practice’ (Bosch & Gascón 1993, 1994), we can now better appreciate the evolution of the research methodologies and the changes related to where the attention is placed. We have chosen the case of Group Theory to present this evolution, as a way to illustrate two different ways of questioning the mathematical content to be taught: a first one based leaving the global structuring of the content untouched; and a second one requiring a complete deconstruction and reconstruction of the knowledge to be taught. This might help better understand the interrelations between the didactic and epistemological assumptions made by teachers and by researchers when designing and experimenting new teaching processes.

FIRST STEP: A WORKSHOP OF PRACTICE ON GROUP THEORY

As said before, our starting point is a research project about the implementation of a new teaching device called Workshops of Practice (WoP) in a Mathematics degree in Barcelona in the 1990s. They were proposed as a way to complement the lectures and problem solving sessions, similar to the 'lab sessions' of the other Natural Sciences degrees (Physics, Chemistry, Biology, Geology). In the new pedagogical organisation proposed, the only modification was to add a weekly 3 hours session devoted to 'practical work' to each subject taught in the degree, each subject being assigned 3 or 4 sessions. What to do in these sessions? What kind of mathematical work appeared as most suitable?

Our proposal was to use the WoP to overcome the two-fold classical organization of university teaching in 'lectures' and 'problem sessions', based on an 'applicationism' vision of mathematics (Barquero et al 2013): students are first introduced to new concepts and mathematical organisations (lectures) and should afterwards apply them to solve a sample of problems of different types. Using the notion of praxeology (Chevallard 2006), we can consider that, in this traditional didactic organisation of university teaching, the first encounter of students with mathematical praxeologies is made through the main elements of their theoretical block (definitions, properties, assumptions, propositions, theorems and proofs). This is then followed by the exploration of the main types of problems conforming the praxeology at stake, using the techniques derived from the theoretical block previously introduced. The direction is always from the theoretical to the practical block of the praxeologies, from the presentation of new concepts and properties to their use to solve problems.

The work proposed to be done in the WoP was to carry out and in-depth study of a single type of problems (or mathematical phenomenon) taken from the course contents, deep enough to let the development of the techniques used and the raise of new theoretical needs. More concretely, each WoP session asked the students to study a given set of similar cases that could initially be solved with the same technique but also required some more or less important variations depending on the specificity of the case. This kind of work had multiple aims. First, students got a first-hand experience of the emergence of new theoretical needs related to the scope of the techniques used and the limits of the type of problems approached. It showed how the development of the practical block of praxeologies motivates the theoretical block. Secondly, it contributed to giving visibility to the 'technical work', which is crucial to mathematics creativity. Finally, it also gave more visibility to the main types of problems that conform the core of the taught subject, thus providing a kind of 'dual description' of the subject (in terms of types of problems instead of notions and theorems).

WoPs were experimented during several academic years in almost all the mathematical subjects of a Mathematics degree, from linear algebra to complex analysis. Their design required a new analysis of the different subjects' contents, usually organised following the logic of the construction of concepts. The example of group theory will help us illustrate it. The subject here is 'Algebra 1' a first year course taught in the second semester, after 'Linear algebra'. Its objective was to introduce the main elements of abstract algebra (groups, rings, fields, morphisms, modular arithmetic, polynomials, etc.) that would be further developed in later courses. An example of a WoP session related to group theory starts with the statement: 'Consider the following groups and establish which ones are isomorphic and which ones are not. When they are not, give reasons. When they are, give a possible isomorphism.' Then a list of 31 groups sorted by their order is given (see figure 1), from order 2 with $(\mathbb{Z}_2,+)$, $(\mathbb{Z}_3^*,\cdot)$, $(\mathbb{Z}_4^*,\cdot)$ and $(\{-1,1\},\cdot)$, till order 8 with $(\mathbb{Z}_8,+)$, $(\mathbb{Z}_4\times\mathbb{Z}_2,+)$ $(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2,+)$ and the dihedral group D4 of the symmetries of a square.

Ordre 2:

a) $(\mathbb{Z}_2,+)$ b) $(\{1,-1\},\cdot)$ c) $(\mathbb{Z}_3^*,\cdot)$ d) $(\mathbb{Z}_4^*,\cdot)$ e) $(\mathbb{Z}_2,\cdot)$

Ordre 3:

a) $(\mathbb{Z}_3,+)$ b) $(A_3,\cdot)$

Ordre 4:

a) $(\mathbb{Z}_4,+)$ b) $(\mathbb{Z}_2\times\mathbb{Z}_2,+)$ c) $(\{1,-1,i,-i\},\cdot)$ d) $(\mathbb{Z}_5^*,\cdot)$ e) $(\mathbb{Z}_8^*,\cdot)$

f) $(K4,\circ)$ amb K4 *el grup 4 de Klein* de les simetries d'un rectangle

g) $(\{f_1,f_2,f_3,f_4\},\circ)$ amb $f_1(x)=x$, $f_2(x)=-x$, $f_3(x)=1/x$ i $f_4(x)=-1/x$

h) $(M4,\cdot)$ amb $M4 = \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}, \begin{pmatrix}-1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&-1\\1&0\end{pmatrix}\right\} \subseteq \mathcal{M}_2(\mathbb{R})$

i) $(B4,\cdot)$ amb $B4 = \{Id, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\} \subseteq \Sigma_4$

Figure 1. Excerpt of the list of finite groups of the workshop of practice

The students had already had some lectures and problem sessions on group theory, with the main definitions, the notion of isomorphism and some examples. The first cases of the workshop thus appeared as non-problematic to them and students could get quickly inside the activity (an unfamiliar situation for most of them). Of course the finding of the isomorphism becomes more difficult as the group order increases. At the same time, having to solve numerous similar cases led many of them to search for properties to skip some steps, for instance by raising the problem of the characterisation of the isomorphism, of the number of groups existing for each order or the properties that are preserved by isomorphism. The very nature of the elements of the set soon seemed to be irrelevant and the strategy of decomposing a group into products of simpler ones sometimes appeared spontaneously.

All these questions, as well as the crucial role played by the group of permutations, were raised during the workshop by the students themselves or by the teacher. Some elements had been introduced in the lectures before the workshop, other were presented in it and treated in detail afterwards. In any case, and even if some of these theoretical elements were frontally introduced by the teacher, it was the technical work done by the students that appeared as the source of new theoretical needs. At the same time, the WoP contributed to make the general praxeological structure of the taught content explicit to the students: what were the 'big type of problems' to approach, what the main techniques to use, what theoretical discourses structured and brought together the whole construction.

THE OFFICIAL RAISON D'ÊTRE OF AN SELF-SUFFICIENT THEORY

From our perspective of researchers in didactics, the aim of the WoPs was to introduce a new device to help students reconstruct the complete local mathematical praxeologies they were supposed to learn. The choice of the lists of problems or cases was based on a previous analysis of the taught subject, its description in terms of amalgams of praxeologies (types of problems, techniques, technological and theoretical discourses) and the search of their *official raison d'être* in the teaching institution, that is types of problems with enough generating power to obtain the main elements of the praxeology.

Given a content such as 'Group Theory' (a whole domain or just a theme, depending on the course considered), what does the search of its *official raison d'être* consists of? It requires determining the main types of tasks solved by this theory in the institution under consideration. For example, in the degree of Mathematics at two universities in Barcelona, Group Theory can appear in different subjects ('Algebra' or 'Algebraic structures', for instance). However, the corresponding content is similar to the one appearing in some standard handbooks:

- Groups, subgroups, homomorphisms.
- Cosets.
- Normality, quotient groups.
- Examples: cyclic, symmetric, alternating, dihedral.
- Isomorphism theorems.
- Group action.
- The Sylow theorems.

After interviewing the teachers in charge of the courses, reading the presentation of contents in the official programmes, consulting the recommended bibliography and the main teaching materials, one notices that the types of tasks concerning Group Theory are *internal* to this theory. The mainly consist in proving that two groups are isomorphic, construction homomorphisms of groups, proving certain properties of given groups or finding examples of groups satisfying certain properties, calculating the index of a subgroup with respect to a given group, proving that a certain subgroup is normal, etc. Therefore, we can claim that, in the institutions under consideration, elementary Group Theory exists for its own sake and is its own raison d'être: to study Group Theory is a goal in itself. Of course, we can claim that, in later courses, students will find non-trivial uses of Group Theory, for example in Galois Theory. This is yet another example of the 'applicationism' vision of mathematics: elementary knowledge first, its application later (Barquero et al 2013).

There are two main didactic facts associated with this state of things. The first one is that students' risk of missing the answers to important questions concerning Group Theory, to the extent that these answers conform the motivation of the theory and that this motivation is hidden. There are elementary questions (for example, why is the structure of group so interesting?) and more sophisticated ones (for instance, why should one be interested in classifying all groups of a given order up to isomorphisms? or why should one be interested in considering cyclic groups, etc.?). All of them are indeed difficult to answer in a non-trivial way without taking the motivation of Group Theory seriously. The second fact is that this 'Euclidean epistemology', as Lakatos defines it, is usually associated with a 'transmissive' didactic strategy placed in the *paradigm of visiting the works* (Chevallard 2015), where mathematical contents are presented to the students as important works to know (even as monuments to visit), without feeling the exigence of justifying or at least showing their importance.

Whether these two facts (unanswered questions about ultimate reasons and the paradigm of visiting works) are undesirable or not is something beyond the scope of a scientific answer, unless they were proved to cause further established inconvenient facts (for example, as regards the utmost aim of Education). In any case, it is still a challenging didactic problem to understand how and to what extent the study of Group Theory can be differently organised in university institutions. The WoPs tried to modify the traditional Euclidean organisation of contents by 'completing' it with a device aiming at articulating problems and theory by generating needs that could motivate new productions. But they do not question the global organisation of the contents, nor their official (and almost always implicit) raison d'être. Furthermore, as in traditional didactic strategies, they keep leaving the responsibility of choosing and posing the problems that are to be studied to the teacher (or to the designer).

Is it possible to go a step further? Can we move from the *paradigm of visiting monuments* to the *paradigm of questioning the world*? What would, in this case, be the transformations needed in the study of Group Theory? Is it possible to find Group Theory as a solution motivated by a certain set of problems? Without prejudging the suitability of any of these two paradigms, the problem seems to be interesting. We will deal with it in the next section.

TOWARDS THE PARADIGM OF QUESTIONING THE WORLD

When moving from *visiting monuments* to *questioning the world*, a new type of didactic analysis is required, focusing not on the *official raison d'être* of Group Theory but on different possible *alternative* ones that could motivate or impel the use of Group Theory as a solution to problematic questions. We are here following a methodology that can seem close to the search for a fundamental situation in terms of the TDS (Brousseau, 1997): What questions can call for the use of the main elements of Group Theory? And what are these elements? What can we do with Group Theory that we cannot do without it? Is there a question that could generate a substantial enough inquiry process so that, at a given moment, Group Theory tools appear to be, if not necessary, at least highly recommendable? Ideally, the answers to all this kind of problems would give rise to what we call a *reference epistemological model* (Barbé et al 2005; Bosch & Gascón 2006), which would provide an alternative reconstruction of Group Theory to help approach problems related to its teaching and learning.

Before sketching some possible initial steps in the epistemological analysis of Group Theory with the aim to illustrate our methodology, let us mention some connected investigations. Some authors have introduced and investigate the so-called FUGS notions, namely, notions that introduce a new *f*ormalism which allows to *u*nify and *g*eneralise (and, consequently, to *s*implify) several mathematical techniques and previous notions (Robert 1998). An example of such a notion would be that of group. Some of these authors advocate the following two thesis, which are directly related to our work. On the one hand, the FUGS character is more frequently attribute to the Mathematics of post-compulsory education than to the Mathematics of compulsory education (Robert 1998). On the other hand, models in terms of situations in the sense of Brousseau (1997) are considered unable to guarantee the genesis of FUGS notions (Robert et Robinet 1996; Robert 1998, Dorier 1995) because of their missing of a 'meta' level.

Concerning the first thesis, we believe that, after specifying the meaning of terms like 'unifying' and 'generalising', everyone would admit that some local praxeologies appearing in basic Mathematics, like numeral systems and measure of quantities, have a strong FUGS status. We also think that the argument supporting the second thesis should be revised, especially when considering 'situation' in the broad sense of alternative epistemological model for the school reconstruction of mathematical pieces of knowledge. Distinctions such as the 'structuralist praxeologies' proposed by Hausberger (2013) might seem more productive.

Let us go back to the epistemological analysis mentioned before. It is well-known that the role played by Group Theory in many fields of Science and Art is linked to the notions of *symmetry* and *invariant* (Weyl 1952): the classification of geometries suggested by Félix Klein (1924/2004), the study of solvability by radicals of a polynomial in Galois Theory, the classification of molecular structures, the classification of bidimensional ornaments,… It is hence shocking that the notions of *symmetry* and *invariant* are missing in practice in the university presentation of elementary Group Theory. Now we wonder: how to integrate these notions at the heart of a possible alternative raison d'être of elementary Group Theory?

In all the examples mentioned, the groups under consideration are sets S_X of bijective maps from a set X to itself, and the invariants are properties (expressed in terms of the base set plus, perhaps, an additional structure) reflected in a certain subgroup. When one considers S_X together with the composition of maps, the axioms of groups appear naturally. In this way, it becomes apparent that the binary operation in groups is not commutative in general. It also becomes clear that a subgroup of a group is just a subset containing the identity map and is closed under the composition and under taking inverse maps, because this is typically the case when one considers the subset of S_X

formed by those maps preserving some property expressed in terms of X. After this, and according to many authors (Freudenthal 1973; Burn 1996; Larsen 2013) it seems reasonable to postulate that elementary Group Theory might appear as a highly recommendable solution of problems set out in this context. The following is a specific example in this direction.

Consider the square C with vertices $V_1 = (-1,1)$, $V_2 = (1,1)$, $V_3 = (1,-1)$ and $V_4 = (-1,-1)$ in the Euclidean plane. Consider the set Sym(C) of symmetries of C, that is to say, the set of isometries of the plane leaving invariant the set of points of the square. It is easy to prove that among the elements of Sym(C) we find the counter-clockwise rotation R with centre (0,0) and right angle, and its powers R^2, R^3 and R^4 which happens to be the identity map $R^4 = I$. It is also easy to check that the reflection T_x (with axis $y = 0$), the reflection T_y (with axis $x = 0$), the reflection T_{13} (with axis the line passing through the vertices V_1 and V_3) and the reflection T_{24} (with axis the line passing through the vertices V_2 and V_4) are also elements of Sym(C). The problem that appears is the following: Is the set $D := \{I, R, R^2, R^3, T_x, T_y, T_{13}, T_{24}\}$ the list of all the elements of Sym(C)?

Let us use elementary Group Theory to prove that D = Sym(C). First, we notice that each element f of Sym(C) induces an element $s(f)$ of the set S_4 of bijections from {1, 2, 3, 4} to itself. Moreover, the map which sends f to $s(f)$ is injective and preserves the composition of maps. For example, we have: $s(I) = I$; $s(R) = (1234)$; $s(R^2) = (13)(24)$; $s(R^3) = (1432)$; $s(T_x) = (14)(23)$; $s(T_y) = (12)(34)$; $s(T_{13}) = (24)$; $s(T_{24}) = (13)$. It is easy to check, from the very definition, that S_4 (in fact S_n for any natural number $n > 1$) is closed under composition, has a neutral element with respect to the composition law, and every element has an inverse with respect to the composition law. Moreover, Sym(C) can be regarded as a subset of S_4 containing the identity map, closed under composition, and closed under taking the inverse map (even if this last condition is not necessary in the case of finite groups).

After these considerations, if in our list D we had forgotten, for instance, to include element R^3, we could claim this list to be incomplete. However, D can be regarded as a subset of S_4 containing the identity map, closed under composition, and closed under taking the inverse map. Thus, we still cannot prove that this list is exhaustive since we might still have missed some element of Sym(C).

The strategy could be a different one. By using elementary Group Theory (more precisely, Lagrange Theorem), we know that the number of elements of Sym(C) is a divisor of the number of elements of S_4. Imagine we have proved that S_4 has $4 \cdot 3 \cdot 2$ elements. Since D is contained in Sym(C) and D has 8 elements, we can say that Sym(C) has at least 8 elements and at most 24 elements. But now, according to Lagrange Theorem, we know that the order of Sym(C) divides 24 (since Sym(C) is isomorphic to a subgroup of S_4) and it is divided by 8 (since D is a subgroup of Sym(C)). Therefore, the only options for the number of elements of Sym(C) is either 8 or 24. But this last option is impossible, since the permutation (23) cannot be in the image of map s. Indeed, the fact that $(23) = s(f)$ for some f in Sym(C), is not compatible with the fact that f preserves the Euclidean distance, since the distance between $f(V_1)$=

V_1 and $f(V_2)= V_3$ is the square root of 8, which, in turn, is different from 2, the distance from V_1 to V_2.

Another problem solved by elementary Group Theory is the classification of the types of symmetries of polygons. Indeed, after the study of the group of symmetries of the regular polygon of *n* vertices and the corresponding lattice of subgroups, one can classify the types of symmetry of the polygons of *n* vertices. For instance, the knowledge of the structure of the group Sym(*C*) (together with the knowledge of the lattice of subgroups) is useful to answer questions concerning the groups of symmetries of convex quadrilaterals, because every such symmetry is also a symmetry of the square. Moreover, this study gives rise to a classification of the quadrilaterals depending on their type of symmetry. This classification corresponds to the structure of the lattice of subgroups of Sym(*C*) and could be more productively exploited in the secondary school Geometry curriculum (Gascón, 2004). Similarly, the study of the group of symmetries of the regular hexagon enables a classification of the possible types of symmetries of the convex hexagons. In general, the group of symmetries of the regular polygon of *n* vertices, together with its lattice of subgroups, can play the role of a *mathematical model* of the *mathematical system* formed by the different types of symmetries of the convex polygons of *n* vertices.

We have presented two problems of the same type which can be set out without explicitly mentioning groups but which can easily be solved with elementary Group Theory. This is good, but it is, by no means, guarantee of success in our research of possible raisons d'être for Group Theory. Others can be found, starting from more extra-mathematical questions, such as the kinship of the indigenous Australian Warlpiri, that also have a structure of a dihedral group (Asher, 2002) or the symmetries of molecules. There are still many open questions left: How promising is this type of problems? For example, is it substantial enough to motivate the study of, for instance, the isomorphism theorems? What about the study of Sylow Theorems? If our motivating type of problems does not give rise to Sylow Theorems, should one consider the possibility of eliminating this from the official program? What if our motivating type of problems requires more knowledge about, for example, geometry, than the one considered in the official program of the degree of Mathematics? This kind of inquiry will soon start questioning the whole domain of Group Theory and, beyond, its relationships with other mathematical domains and the order in which students are supposed to deal with them.

Going further in our inquiry also supposes to consider the important didactics research that has been carried out in the domain (Dubinky et al 1997; Nardi 2000; Lester 2013 among others) and to look at the dimensions of the teaching and learning processes that have been questioned as well as those that have been taken for granted. To what extent, for instance, is the traditional organisation of contents (and the predominance attributed to concepts in detriment of problems) put into question? How are the solid teaching strategies anchored in the pedagogical paradigm of visiting work taken into account by the new inquiry-based teaching proposals?

CONCLUSIONS

We will not pursue the inquiry about possible raisons d'être of Group Theory any further. Our aim was to illustrate, with this specific case, how the research about the Workshop of Practices carried out within the ATD was only based on a partial questioning of the mathematical content taught at university level and of the didactic devices that supported their teaching. A first questioning of the praxeologies that compose the taught mathematical organisation named 'Group Theory' led to identifying some types of problems the in-depth study of which enable students to feel new needs and raise theoretical and practical questions more or less guided by their teacher. However, the proposal of the WoPs took for granted the raisons d'être that the teaching institution – here the university – assigned to the content at stake. And, more importantly, it assumed the fact that its raison d'être should remain implicit, without even formulating the question of its determination, not to speak about its determination by the own students.... Moving from the 'monumentalistic' approach underlying the WoPs to the paradigm of 'questioning the world' requires locating the raisons d'être of the contents that are to be taught and learnt, and the reason of their learning, at the core of the study process. We can even say that the search of this raisons d'être needs to be incorporated in the teaching and learning process itself. It does not appear unreasonable to start an inquiry-based teaching process with the very question: 'What is Group Theory for?'

REFERENCES

Asher, M. (2002). *Mathematics Elsewhere: An Exploration of Ideas Across Cultures.* Princeton: Princeton University Press

Barbé, J., Bosch, M., Espinoza, L., Gascón, J. (2005). Didactic restrictions on the teacher's practice. The case of limits of functions in Spanish High Schools. *Educational Studies in Mathematics*, *59*, 235-268.

Barquero, B., Bosch, M., Gascón, J. (2013). The ecological dimension in the teaching of modelling at university level. *Recherches en Didactique de Mathématiques*, 33(3), 307-338.

Bosch, M., Chevallard, Y. (1999). La sensibilité de l'activité mathématique aux ostensifs. *Recherches en didactique des mathématiques, 19*(1), 77-124

Bosch, M., Gascón, J. (1993). Prácticas en Matemáticas: el trabajo de la técnica. En T. Rojano & L. Puig (Eds.), *Memorias del Tercer Simposio Internacional sobre Investigación en Educación Matemática*, 141-152, Sección Matemática Educativa del CINVESTAV, México DF.

Bosch, M., Gascón, J. (1994). La integración del momento de la técnica en el proceso de estudio de campos de problemas de matemáticas. *Enseñanza de las Ciencias*, 12(3), 314-332.

Bosch, M., Gascón, J. (2006). 25 years of didactic transposition. *ICMI Bulletin*, *58*, 51-64.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Burn, B. (1996). What are the fundamental concepts of group theory? *Educational Studies in Mathematics, 31*, 371-378.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In Bosch, M. (ed.) *Proceedings of the 4th Conference of the European Society for Research in Mathematics Education* (CERME 4). (pp. 21-30) Barcelona: FUNDEMI-IQS.

Chevallard, Y. (2015). Teaching Mathematics in Tomorrow's Society: A Case for an Oncoming Counter Paradigm. In S.J. Cho (ed.), *The Proceedings of the 12th International Congress on Mathematical Education* (pp. 173-187). Dordrecht: Springer.

Chevallard, Y., Bosch, M., Gascón, J. (1997). *Estudiar matemáticas. El eslabón perdido entre la enseñanza y el aprendizaje.* Barcelona, Spain: ICE/Horsori.

Dorier J.-L. (1995). Meta level in the teaching of unifying and generalizing concepts in mathematics, *Educational Mathematical Studies*, 29(2), 175-197.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1997). A Reaction to Burn's "What Are The Fundemental Concepts of Group Theory?". *Educational Studies in Mathematics*, *34*(3), 249-253.

Freudenthal, H. (1973). What groups mean in mathematics and what they should mean in mathematical education. *Developments in mathematical education*, 101-114.

Gascón, J. (2004). Efectos del "autismo temático" sobre el estudio de la Geometría en Secundaria. Parte II. La clasificación de los cuadriláteros convexos. *Suma. Revista sobre la enseñanza y el aprendizaje de las matemáticas*, 45; 41-52

Hausberger, T. (2013). On the concept of (homo)morphism: a key notion in the learning of abstract algebra. In B. Ubuz, C. Haser, M.A. Mariotti (Ed.): *Proceedings of CERME8*. Ankara: Middle East Technical University, 2346-2355.

Klein, F. (2004). Elementary Mathematics from an Advanced Standpoint: Geometry. New York: Dover.

Larsen, S. P. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*,*32*(4), 712-725.

Nardi, E. (2000). Mathematics undergraduates' responses to semantic abbreviations, 'geometric' images and multi-level abstractions in group theory. *Educational Studies in Mathematics 43*, 169-189.

Robert A. (1998). Outils d'analyse des contenus mathématiques à enseigner au lycée et à l'université. *Recherches en Didactique des Mathématique*, 18(2), 139-190.

Robert A., Robinet J. (1996). Prise en compte du méta en didactique des mathématiques, *Recherches en Didactique des Mathématique*, 16(2), 145-176.

Weyl, H. (1952). *Symmetry*, Princeton University Press.